Unit F4

Power series

Introduction

The evaluation of functions is of great importance. If you are dealing with a *polynomial* function, then the calculation of its values is just a matter of arithmetic. For example, if

$$f(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{4}x^4,$$

then

$$f(1) = 1 + \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{4} = \tfrac{13}{12}.$$

On the other hand, the sine function is different. There is no way of calculating most of its values exactly just by the use of arithmetic, but it is important to be able to estimate them accurately because they arise in the solutions of many practical problems.

This unit is concerned with a procedure for calculating approximate values of functions, like the sine function, which cannot be found exactly. In studying this procedure, you will use many of the ideas and results you met in earlier analysis units, especially those relating to sequences and series.

You will see that a certain sequence of polynomials, known as *Taylor polynomials*, can be used to calculate approximate values of functions to any desired degree of accuracy, and that many functions can be represented as a sum of a convergent series of powers of x, known as a *Taylor series*. For example, the polynomial $p(x) = x - x^3/6$ approximates $f(x) = \sin x$ to within 5×10^{-6} for all x in the interval $[0, 0.1]$, and we can represent the sine function by the following convergent series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad \text{for } x \in \mathbb{R}.$$

As an application of these ideas, we show later in the unit how a method based on Taylor series can be used to estimate π to any desired number of decimal places.

1 Taylor polynomials

In this section you will meet the definition of the Taylor polynomial $T_n(x)$ at a point a of a function f, and study several particular functions for which Taylor polynomials appear to provide good approximations.

1.1 What are Taylor polynomials?

Let f be a function defined on an open interval I. Throughout this unit, we assume that a is a particular point of I and seek polynomial functions which provide good approximations to f near the point a.

If f is continuous at a, then the value $f(a)$ is an approximation to the value of $f(x)$ when x is near a, by the definition of continuity; that is,

$$f(x) \approx f(a), \quad \text{for } x \text{ near } a.$$

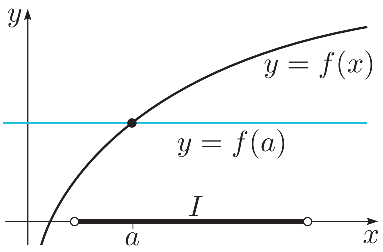


Figure 1 Approximating $f(x)$ by $f(a)$

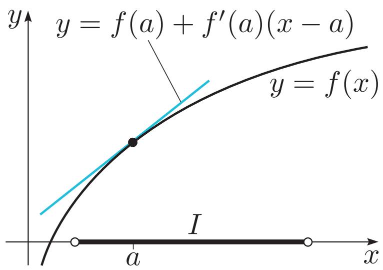


Figure 2 The tangent approximation at a to f

In geometric terms, this means that we can approximate the graph $y = f(x)$ near a by the horizontal line $y = f(a)$ through the point $(a, f(a))$, as shown in Figure 1. Usually this does not give a very good approximation.

However, if the function f is differentiable on I, then we can obtain what is usually a better approximation by using the tangent at $(a, f(a))$ instead of the horizontal line, as shown in Figure 2. We can think of the tangent at $(a, f(a))$ as the *line of best approximation* to the graph near a. The tangent to the graph at $(a, f(a))$ has equation

$$\frac{y - f(a)}{x - a} = f'(a), \quad \text{that is,} \quad y = f(a) + f'(a)(x - a).$$

So, for x near a, we can write

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is called the **tangent approximation** at a to f.

Note that the function f and the approximating linear function

$$x \longmapsto f(a) + f'(a)(x - a)$$

have the same value at a and the same first derivative at a (that is, their graphs have the same gradient at a), so this gives a better approximation to f near a than the constant function $f(a)$ when the gradient of the graph is non-zero.

Worked Exercise F40

Determine the tangent approximation to the function $f(x) = e^x$ at the point 0.

Solution

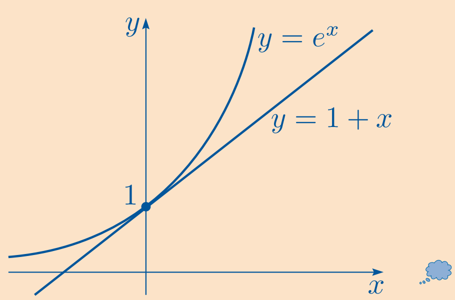
Here $a = 0$ and

$$f(x) = e^x, \quad f(0) = 1$$
$$f'(x) = e^x, \quad f'(0) = 1.$$

Hence the tangent approximation to f at 0 is

$$f(x) \approx f(0) + f'(0)(x - 0) = 1 + x.$$

This is illustrated in the graph below.



Exercise F55

Determine the tangent approximation to each of the following functions f at the given point a.

(a) $f(x) = e^x, \quad a = 2.$ (b) $f(x) = \cos x, \quad a = 0.$

So far we have seen two approximations to $f(x)$ for x near a:

$$f(x) \approx f(a) \quad \text{(a constant function)},$$
$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{(a linear function)}.$$

If the function f is twice differentiable on I, then we can consider the quadratic function

$$p(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2,$$

which is chosen to satisfy $p(a) = f(a)$, $p'(a) = f'(a)$ and $p''(a) = f''(a)$, as you can check by differentiating p to give

$$p'(x) = f'(a) + f''(a)(x-a), \quad \text{and} \quad p''(x) = f''(a).$$

It is plausible that, for x near a,

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2$$

will usually be a better approximation to f near a than the constant or linear approximations. More generally, if the function f is n-times differentiable on I, then we can find a polynomial of degree n whose value at a and first n derivatives at a are equal to those of f, and this polynomial will usually provide an even better approximation.

Definition

Let f be n-times differentiable on an open interval containing the point a. Then the **Taylor polynomial of degree n at a for f** is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

that is,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^n.$$

Remarks

1. Of course, the Taylor polynomial depends on the point a and the function f as well as on n and x, but for brevity our notation $T_n(x)$ does not reflect this. Usually a and f will be clear from the context.
2. The coefficients in the definition of T_n have been chosen so that
$$T_n(a) = f(a), \quad T_n'(a) = f'(a), \quad \ldots, \quad T_n^{(n)}(a) = f^{(n)}(a).$$
Thus the functions f and T_n have the same value at a and have equal derivatives at a for all orders up to and including n, in the same way as for the linear and quadratic functions above. Indeed, $T_0(x)$, $T_1(x)$ and $T_2(x)$ are respectively the constant, linear and quadratic approximations at a to f discussed previously.
3. It follows from the definition that Taylor polynomials for successive values of n satisfy the recurrence relation
$$T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad n \geq 1.$$
4. A Taylor polynomial has degree n if it is based on derivatives of f up to order n. This use of the word 'degree' differs from the usual definition of the degree of a polynomial, which is the largest exponent in the polynomial.
5. Some texts refer to a Taylor polynomial *about* a instead of *at* a.

Worked Exercise F41

Determine the Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ at the following points a for the function $f(x) = \sin x$.

(a) $a = 0$ (b) $a = \pi/2$

Solution

To find the Taylor polynomial of degree n at a, we need to work out the first n derivatives of f at a.

We have

$$\begin{aligned}
f(x) &= \sin x, & f(0) &= 0, & f(\pi/2) &= 1\\
f'(x) &= \cos x, & f'(0) &= 1, & f'(\pi/2) &= 0\\
f''(x) &= -\sin x, & f''(0) &= 0, & f''(\pi/2) &= -1\\
f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1, & f^{(3)}(\pi/2) &= 0.
\end{aligned}$$

It is convenient to use the recurrence relation for $T_n(x)$ when determining Taylor polynomials for successive values of n.

(a) Hence, at $a = 0$,

$$T_1(x) = f(0) + f'(0)x = x$$
$$T_2(x) = T_1(x) + \frac{f''(0)}{2!}x^2 = x$$
$$T_3(x) = T_2(x) + \frac{f^{(3)}(0)}{3!}x^3 = x - \tfrac{1}{6}x^3.$$

Note that $T_2(x)$, the Taylor polynomial of degree 2 at 0, is a polynomial of degree 1 because $f''(0) = 0$.

(b) At $a = \pi/2$ we have

$$T_1(x) = f(\pi/2) + f'(\pi/2)(x - \pi/2) = 1$$
$$\begin{aligned} T_2(x) &= T_1(x) + \frac{f''(\pi/2)}{2!}(x - \pi/2)^2 \\ &= 1 - \tfrac{1}{2}(x - \pi/2)^2 \end{aligned}$$
$$\begin{aligned} T_3(x) &= T_2(x) + \frac{f^{(3)}(\pi/2)}{3!}(x - \pi/2)^3 \\ &= 1 - \tfrac{1}{2}(x - \pi/2)^2. \end{aligned}$$

We do not usually multiply out brackets in such Taylor polynomials, since that would make the results less clear.

Exercise F56

Determine the Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ for each of the following functions f at the given point a.

(a) $f(x) = e^x$, $a = 2$. (b) $f(x) = \cos x$, $a = 0$.

Exercise F57

Determine the Taylor polynomial of degree 4 for each of the following functions f at the given point a.

(a) $f(x) = \log(1 + x)$, $a = 0$. (b) $f(x) = \sin x$, $a = \pi/4$.

(c) $f(x) = 1 + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{4}x^4$, $a = 0$.

Exercise F58

Let T_3 be the Taylor polynomial of degree 3 at 0 for $f(x) = \sin x$, as calculated in Worked Exercise F41(a). Use a calculator to show that

$$|\sin(0.1) - T_3(0.1)| < 1 \times 10^{-7}.$$

(Remember to set the calculator to use angles in radians.)

1.2 Approximation by Taylor polynomials

We now look at two specific functions in order to investigate the assertion that Taylor polynomials provide good approximations for a large class of functions.

The function $f(x) = \sin x$

In Worked Exercise F41(a) we found the Taylor polynomials $T_1(x)$, $T_2(x)$ and $T_3(x)$ for the function $f(x) = \sin x$ at the point $a = 0$. By calculating higher derivatives of f at 0, we can show that the Taylor polynomials of degrees $1, 2, \ldots, 8$ at 0 for f are as follows.

$$T_1(x) = T_2(x) = x, \qquad T_3(x) = T_4(x) = x - \frac{x^3}{3!},$$
$$T_5(x) = T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad T_7(x) = T_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

The graphs in Figure 3 illustrate how the approximation to $f(x)$ given by $T_n(x)$ improves as n increases.

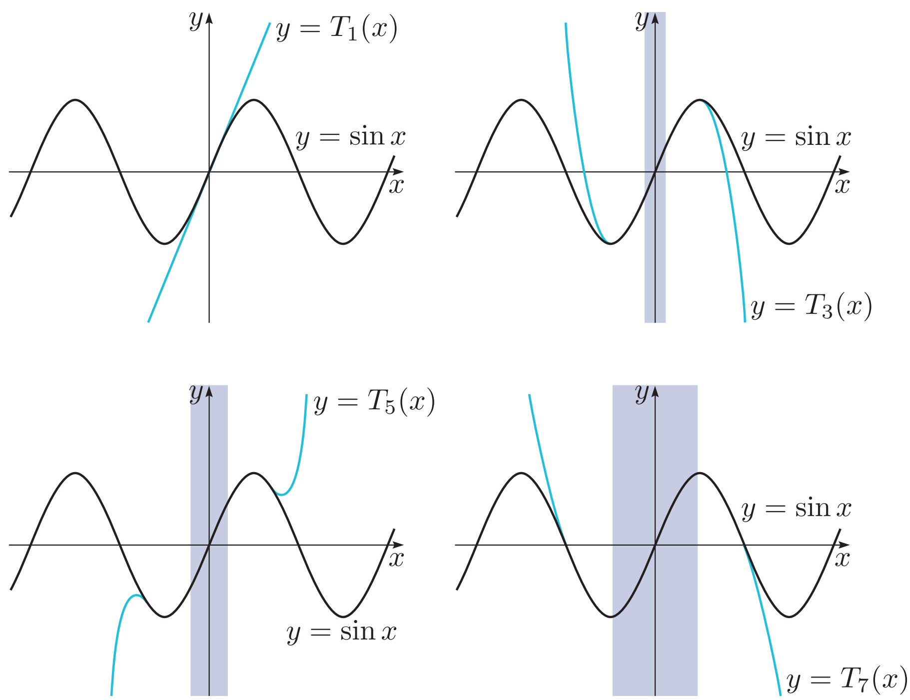


Figure 3 The graphs of Taylor polynomials at 0 for $f(x) = \sin x$

For example, the graph of T_5 appears to be very close to the graph of the sine function over the interval $(-\pi/2, \pi/2)$, so $T_5(x)$ seems to be a good approximation to $\sin x$ in this interval.

It also appears that, as the degree of the Taylor polynomial increases, the interval over which its graph is a good approximation to that of the sine function becomes longer. For instance, in the graphs in Figure 3, the shaded area covers the interval of the x-axis on which the Taylor polynomial $T_n(x)$ agrees with $\sin x$ to three decimal places.

Worked Exercise F42

Determine the Taylor polynomial of degree n at 0 for the function $f(x) = \sin x$.

Solution

We have

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0 \\ f'(x) &= \cos x, & f'(0) &= 1 \\ f''(x) &= -\sin x, & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x, & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0 \end{aligned}$$

and in general, for $k = 0, 1, 2, \ldots,$

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

Hence, for $m = 0, 1, 2, \ldots,$

$$\begin{aligned} T_{2m+1}(x) &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \end{aligned}$$

and $T_{2m+2}(x) = T_{2m+1}(x)$.

So if n takes either of the values $2m+1$ or $2m+2$, then we have

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}.$$

The function $f(x) = 1/(1-x)$

By repeated differentiation of $f(x) = 1/(1-x)$, we can verify that

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}, \quad \text{for } k \in \mathbb{N}.$$

Thus in particular $f^{(k)}(0) = k!$, so the Taylor polynomial of degree n at 0 for f is

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n. \end{aligned}$$

Figure 4 shows the graphs of the Taylor polynomials of degrees 1, 2, 4 and 7 at 0 for f.

The graphs show that the nature of the approximation is very different from that in the previous example. For the sine function, the interval over which the approximation is good seems to expand indefinitely as the degree of the polynomial increases. For $f(x) = 1/(1-x)$, however, the interval of good approximation always seems to be contained in the interval $(-1, 1)$.

For this function f, the Taylor polynomials $T_n(x)$ at 0 are the nth partial sums of the geometric series $\sum_{n=0}^{\infty} x^n$. This series converges with sum $1/(1-x)$ for $|x| < 1$, and diverges for $|x| \geq 1$, as you saw in Theorem D24 in Unit D3 *Series*. Thus, if $|x| < 1$, then

$$T_n(x) \to f(x) \text{ as } n \to \infty.$$

So in this example we can prove that the polynomials $T_n(x)$ provide better and better approximations to $f(x)$ as n increases, but only if $|x| < 1$. For $|x| \geq 1$, the sequence $(T_n(x))$ does not converge, so increasing the value of n does not in general give a better approximation to $f(x)$.

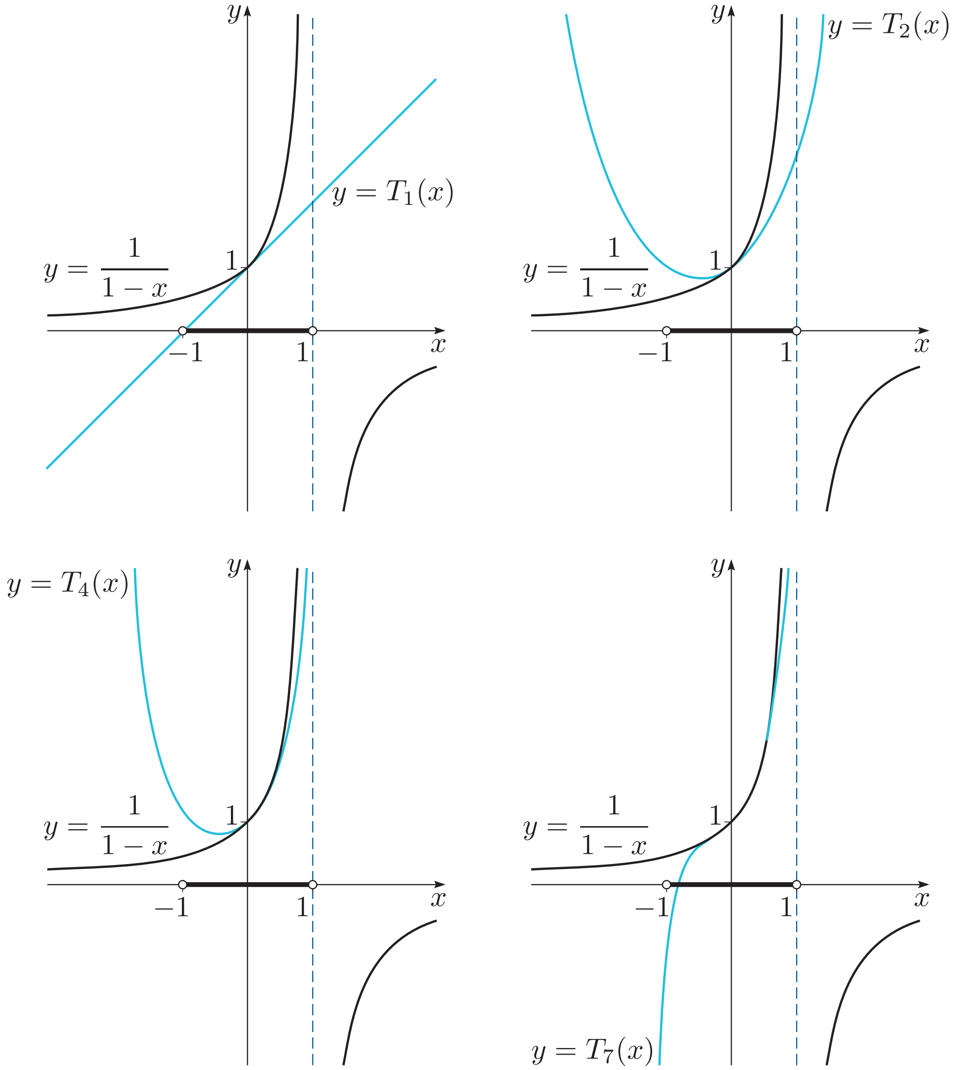


Figure 4 The graphs of Taylor polynomials at 0 for $f(x) = 1/(1-x)$

In later sections we often need general formulas for certain key Taylor polynomials at the point $a = 0$; we ask you to find these in the next exercise.

Exercise F59

Determine the Taylor polynomial of degree n at 0 for each of the following functions.

(a) $f(x) = e^x$ (b) $f(x) = \log(1+x)$ (c) $f(x) = \cos x$

2 Taylor's Theorem

In this section you will investigate how closely the Taylor polynomials of a function f approximate f and see that, for many functions, f has a representation at the point $x = a$ of the form

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} a_n(x-a)^n.$$

We then say that f is the *sum function* of a *power series*.

2.1 Taylor's Theorem

In Section 1 we showed how to find the Taylor polynomial $T_n(x)$ of degree n at the point a for a function f. This polynomial and its first n derivatives agree with f and its first n derivatives at a, and for larger values of n the polynomial appears to approximate f at points near a. The following fundamental result gives a formula for the error involved in this approximation.

Theorem F63 Taylor's Theorem

Let the function f be $(n+1)$-times differentiable on an open interval containing the points a and x. Then

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \\ &= T_n(x) + R_n(x), \end{aligned}$$

where $T_n(x)$ is the Taylor polynomial of degree n at a for f and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between a and x.

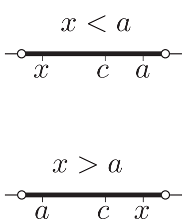


Figure 5 The points a, x and c in Taylor's Theorem

Remarks

1. Figure 5 shows the possible relative positions of the points a, x and c.
2. Since Taylor's Theorem can be expressed in the form

   $$f(x) = T_n(x) + R_n(x),$$

   we say that $R_n(x)$ is a **remainder term**, or **error term**, involved in approximating $f(x)$ by $T_n(x)$. The formula for $R_n(x)$ involves an 'unknown number' c, so it does not specify the remainder term $R_n(x)$ exactly. Nevertheless, we can often use it to show that $T_n(x)$ is a good approximation to $f(x)$. Note that, as with the notation $T_n(x)$, our notation $R_n(x)$ does not reflect the fact that the remainder also depends on both f and a.

3. When $n = 0$, Taylor's Theorem reduces to

$$f(x) = f(a) + f'(c)(x - a),$$

for some c between a and x; that is,

$$\frac{f(x) - f(a)}{x - a} = f'(c),$$

for some c between a and x. But this is just the Mean Value Theorem that you met in Subsection 4.1 of Unit F2 *Differentiation* (see Figure 6). Thus Taylor's Theorem can be considered as a generalisation of the Mean Value Theorem.

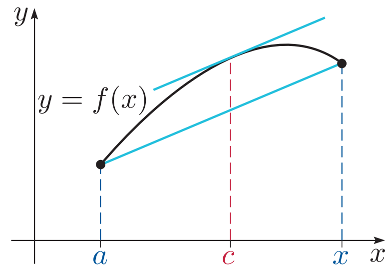


Figure 6 The Mean Value Theorem

4. The form of the remainder $R_n(x)$ given in Taylor's Theorem is actually due to Lagrange. There are other forms, due to Taylor and Cauchy, and also the following neat formula which can be derived by repeated integration by parts (we do not prove this here):

$$R_n(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t)\, dt.$$

Note that this form of the remainder does not involve any 'unknown numbers'.

Brook Taylor (1685–1731) was an English mathematician who in 1715 published a slim volume entitled *Methodus incrementorum* (Method of Increments) which included the theorem that now bears his name. Taylor was the first to publish the theorem but he was not the first to discover it. At least five mathematicians anticipated him: James Gregory (1671), Leibniz (1670s), Newton (1691), Johann Bernoulli (1694) and de Moivre (1708). However, Taylor was the first to have appreciated the fundamental significance of the result.

The first explicit expression for the remainder term in Taylor's theorem was provided by Joseph-Louis Lagrange in 1797 in his *Théorie des fonctions analytiques* (Theory of analytic functions), the text in which he attempted to provide a sound foundation for calculus by reducing it to algebra and developing it on the basis of Taylor's Theorem.

Taylor was an accomplished musician and artist, and in his book on linear perspective, which was also first published in 1715, with an improved version in 1719, he enriched the theory of perspective in many respects.

Brook Taylor

In Subsection 4.1 of Unit F2 we proved the Mean Value Theorem using Rolle's Theorem, which you also met in Unit F2. This says that, if f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there is a point c with $a < c < b$ such that $f'(c) = 0$. We now use Rolle's Theorem to prove Taylor's Theorem. If you are short of time, then you may prefer to skim read this proof.

Proof of Taylor's Theorem In the proof we assume that $x > a$; the proof for $x < a$ is similar.

We consider the function

$$h(t) = f(t) - T_n(t) - A(t-a)^{n+1}, \tag{1}$$

where T_n is the Taylor polynomial of degree n at a for f, and A is a constant chosen so that

$$h(x) = 0. \tag{2}$$

Now, by the definition of T_n,

$$f(a) = T_n(a), \quad f'(a) = T_n'(a), \quad \ldots, \quad f^{(n)}(a) = T_n^{(n)}(a),$$

so

$$h(a) = 0, \quad h'(a) = 0, \quad \ldots, \quad h^{(n)}(a) = 0.$$

Thus the function h is continuous and differentiable on an open interval containing a and x, and $h(a) = 0 = h(x)$. So, by Rolle's Theorem applied to h on the interval $[a, x]$, there is a number c_1 between a and x for which

$$h'(c_1) = 0.$$

Similarly, the function h' is continuous and differentiable on an open interval containing a and c_1, and $h'(a) = 0 = h'(c_1)$. Hence, by Rolle's Theorem applied to h' on the interval $[a, c_1]$, there is a number c_2 between a and c_1 for which

$$h''(c_2) = 0.$$

Applying Rolle's Theorem successively to the functions

$$h'',\ h^{(3)},\ \ldots,\ h^{(n)},$$

on the intervals

$$[a, c_2],\ [a, c_3],\ \ldots,\ [a, c_n], \quad \text{where } c_2 > c_3 > \cdots > c_n > a,$$

we deduce that there is a number c between a and c_n for which

$$h^{(n+1)}(c) = 0. \tag{3}$$

These points are illustrated in Figure 7.

Figure 7 The points a, c, c_1, c_2, $\ldots$, c_n and x

By repeatedly differentiating equation (1), we obtain

$$h^{(n+1)}(t) = f^{(n+1)}(t) - A(n+1)!. \tag{4}$$

Note that $T_n^{(n+1)}(t) = 0$ since T_n is a polynomial of degree at most n and differentiating such a polynomial n times gives a constant.

From equations (3) and (4), we deduce that

$$f^{(n+1)}(c) - A(n+1)! = 0, \quad \text{so} \quad A = \frac{f^{(n+1)}(c)}{(n+1)!}. \tag{5}$$

Finally, it follows from equations (1), (2) and (5) that

$$f(x) = T_n(x) + A(x-a)^{n+1} = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

as required. ■

Exercise F60

By applying Taylor's Theorem with $n = 3$ to the function $f(x) = \cos x$ at $a = 0$, prove that, for $x \neq 0$,

$$\cos x = 1 - \tfrac{1}{2}x^2 + \frac{\cos c}{4!}\,x^4,$$

where c lies between 0 and x.

(The Taylor polynomial of degree n at 0 for $\cos x$ was found in Exercise F59(c)).

The conclusion of Exercise F60 can be restated as

$$\cos x - (1 - \tfrac{1}{2}x^2) = \frac{\cos c}{4!}x^4,$$

where c lies between 0 and x. Here we do not know the exact value of c, but we do know that $|\cos c| \leq 1$. Thus we can deduce that

$$\left|\cos x - (1 - \tfrac{1}{2}x^2)\right| = \frac{|\cos c|}{4!}|x|^4 \leq \frac{|x|^4}{4!}.$$

In this way, we obtain an explicit **remainder estimate**, or **error bound**, for the approximation of $\cos x$ by $1 - \frac{1}{2}x^2$, which is small when x is near 0.

In general, we can obtain such an estimate for $|f(x) - T_n(x)| = |R_n(x)|$ provided that we have an estimate for $|f^{(n+1)}(c)|$ which is valid for all c between a and x. The following strategy sets out this process.

Strategy F11

To show that the Taylor polynomial T_n at a for f approximates f to a certain accuracy at a point $x \neq a$, do the following.

1. Obtain a formula for $f^{(n+1)}$.
2. Determine a number M such that

   $$|f^{(n+1)}(c)| \leq M, \quad \text{for all } c \text{ between } a \text{ and } x.$$

3. Write down and simplify the remainder estimate

   $$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Note that in step 2 we can use any convenient value for M, preferably not too large. Sometimes we can determine the maximum value of $|f^{(n+1)}(c)|$ for c in the closed interval with endpoints a and x, and often this maximum value is taken when c is equal to either a or x. Usually, however, any 'good enough' upper bound for $|f^{(n+1)}(c)|$ will do.

Worked Exercise F43

(a) Write down the Taylor polynomial $T_3(x)$ at $a = 0$ for $f(x) = \sin x$.

(b) Use Taylor's Theorem to show that $|\sin(0.1) - T_3(0.1)| < 5 \times 10^{-6}$.

(c) Hence calculate $\sin(0.1)$ to four decimal places.

Solution

(a) For $f(x) = \sin x$ and $a = 0$, we have

$$T_3(x) = x - \tfrac{1}{6}x^3.$$

This expression for $T_3(x)$ was derived in Worked Exercise F41(a).

(b) We use Strategy F11 with $a = 0$, $x = 0.1$ and $n = 3$.

1. First, $f^{(4)}(x) = \sin x$.
2. Thus

$$|f^{(4)}(c)| = |\sin c| \leq 1, \quad \text{for } c \in [0, 0.1],$$

so we can take $M = 1$.

With care a smaller value for M can be obtained. For example, using the Sine Inequality (Theorem D45 in Unit D4 *Continuity*), we have

$$|\sin c| \leq |c| \leq 0.1, \quad \text{for } c \in [0, 0.1].$$

However, here we take $M = 1$.

3. Using the remainder estimate $\dfrac{M}{(n+1)!}|x-a|^{n+1}$, we therefore obtain

$$\begin{aligned} |\sin(0.1) - T_3(0.1)| &= |R_3(0.1)| \\ &\leq \frac{M}{(3+1)!}|x-a|^{3+1} \\ &= \frac{1}{4!}|0.1-0|^4 \\ &= 0.000\,004\,1\overline{6} \\ &< 5 \times 10^{-6}, \end{aligned}$$

as required.

(c) By part (a),

$$\begin{aligned} T_3(0.1) &= 0.1 - \tfrac{1}{6} \times 10^{-3} \\ &= 0.1 - 0.000\,166\,66\ldots = 0.099\,833\,33\ldots. \end{aligned}$$

By part (b),

$$|\sin(0.1) - T_3(0.1)| = |R_3(0.1)| < 5 \times 10^{-6}.$$

Hence

$$0.099\,828\,33\ldots < \sin(0.1) < 0.099\,838\,33\ldots,$$

so

$$\sin(0.1) = 0.0998 \quad \text{(to 4 d.p.)}.$$

Exercise F61

(a) Write down the Taylor polynomial $T_2(x)$ at $a = 0$ for $f(x) = \log(1 + x)$, using the solution to Exercise F59(b).

(b) Use Taylor's Theorem to show that $|\log(1.02) - T_2(0.02)| < 3 \times 10^{-6}$.

(c) Hence calculate $\log(1.02)$ to four decimal places.

Our next strategy shows how to use Taylor's Theorem to obtain an approximation to f which holds at all points of an interval.

Strategy F12

To show that the Taylor polynomial T_n at a for f approximates f to a certain accuracy throughout an interval I of the form $[a, a + r]$, $[a - r, a]$ or $[a - r, a + r]$, where $r > 0$, do the following.

1. Obtain a formula for $f^{(n+1)}$.
2. Determine a number M such that
$$|f^{(n+1)}(c)| \leq M, \quad \text{for all } c \in I.$$
3. Write down and simplify the remainder estimate
$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n + 1)!} r^{n+1}, \quad \text{for all } x \in I.$$

Strategy F12 is obtained from Strategy F11 by replacing $|x - a|$ by r, since $|x - a| \leq r$ for all values of x in the interval I.

By applying Strategy F12 to the situation in Worked Exercise F43, we can show that the remainder estimate at the point $x = 0.1$ in fact holds over the whole interval $[0, 0.1]$. Here we have $r = 0.1$, $M = 1$ and $n = 3$, so

$$|\sin x - T_3(x)| \leq \frac{1}{4!} 0.1^4 < 5 \times 10^{-6}, \quad \text{for all } x \in [0, 0.1].$$

Here is another worked exercise.

Worked Exercise F44

(a) Calculate the Taylor polynomial $T_3(x)$ at 1 for $f(x) = 1/(x+2)$.

(b) Show that $T_3(x)$ approximates $f(x)$ with an error less than 5×10^{-3} on the interval $[1, 2]$.

Solution

(a) For this function,

$$\begin{aligned} f(x) &= 1/(x+2), & f(1) &= 1/3 \\ f'(x) &= -1/(x+2)^2, & f'(1) &= -1/9 \\ f''(x) &= 2/(x+2)^3, & f''(1) &= 2/27 \\ f^{(3)}(x) &= -6/(x+2)^4, & f^{(3)}(1) &= -2/27. \end{aligned}$$

Hence the Taylor polynomial of degree 3 at 1 for f is

$$T_3(x) = \tfrac{1}{3} - \tfrac{1}{9}(x-1) + \tfrac{1}{27}(x-1)^2 - \tfrac{1}{81}(x-1)^3.$$

(b) We use Strategy F12 with $I = [1, 2]$, $a = 1$, $r = 1$ and $n = 3$.

1. First, $f^{(4)}(x) = \dfrac{24}{(x+2)^5}$.
2. Thus
$$|f^{(4)}(c)| = \frac{24}{(c+2)^5} \leq \frac{24}{3^5}, \quad \text{for } c \in [1, 2],$$
so we can take $M = 24/3^5$.
Here we have used the fact that, since $c \geq 1$, we have $c + 2 \geq 1 + 2 = 3$.
3. Using the remainder estimate $\dfrac{M}{(n+1)!}\, r^{n+1}$, we obtain
$$\begin{aligned} |f(x) - T_3(x)| &= |R_3(x)| \\ &\leq \frac{M}{(3+1)!}\, r^{3+1} \\ &= \frac{1}{4!} \times \frac{24}{3^5} \times 1^4 \\ &= \frac{1}{3^5} = 0.0041\ldots, \quad \text{for } x \in [1, 2]. \end{aligned}$$

Thus $T_3(x)$ approximates $f(x)$ with an error less than 5×10^{-3} on $[1, 2]$.

Exercise F62

(a) Calculate the Taylor polynomial $T_4(x)$ at π for the function $f(x) = \cos x$.

(b) Show that $T_4(x)$ approximates $f(x)$ with an error less than 3×10^{-3} on the interval $[3\pi/4, 5\pi/4]$.

2.2 Taylor series

From Taylor's Theorem we know that if a function f can be differentiated as often as we want on an open interval containing the points a and x, then

$$f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x),$$

for $n = 0, 1, 2, \ldots,$ where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between a and x. So if the error terms $R_n(x)$ tend to zero, then the Taylor polynomials $T_n(x)$ tend to $f(x)$ as n tends to infinity, and we can write f as an infinite series. Thus we have the following result.

Theorem F64

Let f have derivatives of all orders on an open interval containing the points a and x. If

$$R_n(x) \to 0 \text{ as } n \to \infty,$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that, for $x = a$ and $n = 0$, the series in the statement of Theorem F64 involves the expression 0^0. By convention, we take $0^0 = 1$ in such series.

It follows from Theorem F64 that, if $R_n(x) \to 0$ as $n \to \infty$, then we can express $f(x)$ as a series whose terms involve powers of $(x - a)$.

Definition

Let f have derivatives of all orders at the point a. The **Taylor series at a for f** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

If x is a point for which the Taylor series for f has the sum $f(x)$ given in the statement of Theorem F64 (that is, when $R_n(x) \to 0$ as $n \to \infty$), then we say that the Taylor series is **valid** at the point x. Any set of values of x for which a Taylor series is valid is called a **range of validity** for the Taylor series. On any such range of validity, the function f is the **sum function** of the Taylor series.

We can use Theorem F64 to obtain the following basic Taylor series. In each case we have indicated the largest possible range of validity.

Theorem F65 Basic Taylor series at 0

(a) $$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

(b) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \text{for } x \in \mathbb{R}.$$

(c) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \text{for } x \in \mathbb{R}.$$

(d) $$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for } x \in \mathbb{R}.$$

(e) $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad \text{for } -1 < x \leq 1.$$

Remarks

1. Taking $x = 1$ in Theorem F65(e), we obtain the unexpected sum
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \log 2.$$
This series is known as the **alternating harmonic series**.
2. In Subsection 4.1 of Unit D3 we defined the exponential function as
$$e^x = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}, & x \geq 0, \\ (e^{-x})^{-1}, & x < 0. \end{cases}$$
Theorem F65(d) shows that e^x is the sum function of the series $\sum_{n=0}^{\infty} x^n/n!$ for *all* x, not just for $x \geq 0$. Note that some texts define $e^x = \exp(x)$ using this power series.
3. In this module, $\sin x$ and $\cos x$ are defined in terms of a right-angled triangle. Theorem F65 shows that $\sin x$ and $\cos x$ can be represented by power series for all $x \in \mathbb{R}$, and some texts use these series to define $\sin x$ and $\cos x$ in a way that does not depend on geometric ideas.

Proof of Theorem F65

(a) Let $f(x) = 1/(1-x)$. You saw in Subsection 1.2 that the Taylor polynomial of degree n at 0 for f is
$$T_n(x) = 1 + x + \cdots + x^n = \sum_{k=0}^{n} x^k.$$

Now $1 + x + x^2 + \cdots$ is a geometric series with initial term 1 and common ratio x, which has sum $1/(1-x)$ for $|x| < 1$, as you saw in Theorem D24 in Unit D3. Thus the result follows.

(b) Let $f(x) = \sin x$. You saw in Worked Exercise F42 that the Taylor polynomial of degree n at 0 for f is

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

where $n = 2m+1$ or $n = 2m+2$.

By Taylor's Theorem, we have

$$\sin x = T_n(x) + R_n(x), \quad \text{where } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1},$$

for some c between 0 and x. Since

$$f^{(n+1)}(x) = \pm \sin x \text{ or } \pm \cos x,$$

we have $|f^{(n+1)}(c)| \leq 1$, so

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

Here we have used the Squeeze Rule for null sequences and the fact that $(x^n/n!)$ is a basic null sequence; see Subsection 2.3 of Unit D2 *Sequences*.

Hence the result follows.

(c) The proof of part (c) is similar to that of part (b), so we omit the details.

(d) Let $f(x) = e^x$. You saw in the solution to Exercise F59(a) that the Taylor polynomial of degree n at 0 for f is

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

By Taylor's Theorem, we have

$$e^x = T_n(x) + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1} = e^c \frac{x^{n+1}}{(n+1)!},$$

for some c between 0 and x. Now $e^c \leq e^{|x|}$ and $(x^{n+1}/(n+1)!)$ is a null sequence.

The value of c depends on n, but it always lies between 0 and x.

Hence $R_n(x) \to 0$ as $n \to \infty$, by the Squeeze Rule for null sequences, so the result follows.

(e) Let $f(x) = \log(1+x)$. You saw in the solution to Exercise F59(b) that the Taylor polynomial of degree n at 0 for f for $n \geq 1$ is

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k}.$$

By Taylor's Theorem, we have

$$\log(1+x) = T_n(x) + R_n(x), \quad \text{where } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1},$$

for some c between 0 and x.

Now suppose that $0 < x \leq 1$. Since, by the solution to Exercise F59(b),

$$f^{(n+1)}(x) = \frac{(-1)^{n+2}n!}{(1+x)^{n+1}},$$

we have

$$\begin{aligned} |R_n(x)| &= \frac{1}{(n+1)!} \times \frac{n!}{(1+c)^{n+1}}\, |x|^{n+1} \\ &= \frac{|x|^{n+1}}{(n+1)(1+c)^{n+1}} \leq \frac{1}{n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad \text{for } 0 \leq x \leq 1.$$

Here we have used the fact that $0 < c < x \leq 1$, so

$$|x|^{n+1} \leq 1 \quad \text{and} \quad 1 + c > 1.$$

We can include $x = 0$ in the range of validity because both $\log(1+x)$ and the sum function are zero when $x = 0$.

The proof that this Taylor series is also valid for $-1 < x < 0$ does not follow from the above form of $R_n(x)$; we ask you to prove it later, in Exercise F72. ■

3 Convergence of power series

So far in this unit you have seen how a function f can often be approximated near a point a by means of a Taylor series, which is an infinite sum of powers of $(x - a)$. In this section you will study the behaviour of such *power series* in their own right, and consider functions which are *defined* by power series.

3.1 Radius of convergence

We begin with a formal definition of a power series.

Definitions

Let $a \in \mathbb{R}$, $x \in \mathbb{R}$ and $a_n \in \mathbb{R}$, $n = 0, 1, 2, \ldots$. Then the expression

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

is called a **power series at a in x**, with **coefficients** a_n. We call a the **centre** of the power series.

Notice that in the definition of a power series, we think of a as a constant and x as a variable.

In Section 2 you saw that certain standard functions can be expressed as the sum functions of their Taylor series; for example,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1,$$

and

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad \text{for } -1 < x \le 1,$$

and both these series are power series at 0 in x. All Taylor series are examples of power series.

On the other hand, we can consider power series in their own right and use these to *define* functions. For example, you saw in Subsection 2.2 that we could have defined the exponential function by the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R}).$$

Another example is the *Bessel function*

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x/2)^{2n}}{(n!)^2} \quad (x \in \mathbb{R}),$$

which arises in connection with the vibration of a circular drum.

In each of the above examples (where $a = 0$), the power series converges on an interval with centre a. The next result shows that this property is true for all power series; it is illustrated in Figure 8. We give the proof of this result in Subsection 3.2.

Figure 8 The radius of convergence

Theorem F66 Radius of Convergence Theorem

For a given power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$, exactly one of the following possibilities occurs.

(a) The series converges only for $x = a$.

(b) The series converges for all x.

(c) There is a number $R > 0$ such that

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges if } |x-a| < R$$

and

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ diverges if } |x-a| > R.$$

Moreover, in parts (a), (b) and (c) the series converges absolutely on the specified sets of convergence.

Although it is usually sufficient to know that a series is convergent, the stronger result about absolute convergence is sometimes useful (for example, we will use it when we prove the Differentiation Rule in Subsection 4.2). Remember that a series $\sum a_n$ converges absolutely if the series $\sum |a_n|$ converges. We showed in Theorem D34 in Unit D3 that every absolutely convergent series is convergent.

You have already met the following examples of the three possibilities in Theorem F66 (see Theorem F65(d) for (b) and Unit D3 for (a) and (c)):

(a) $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$ converges only for $x = 0$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x

(c) $\displaystyle\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| > 1$, so $R = 1$.

The positive number R in Theorem F66(c) is called the **radius of convergence** of the power series because the power series converges at those points whose distance from the centre a is less than R, and diverges at those points whose distance from a is greater than R. (Power series can also be defined for complex variables, in which case the points whose distance from the centre is less than R form a disc of radius R.) We extend the definition of the radius of convergence to the cases of Theorem F66(a) and (b) by writing

$R = 0$ if the power series converges only for $x = a$

and

$R = \infty$ if the power series converges for all x.

In this last case R is used as a symbol, not a real number.

Theorem F66(c) makes no assertion about the behaviour of the power series at the endpoints of the interval $(a - R, a + R)$; in fact, a power series may converge at both endpoints, neither endpoint or exactly one endpoint, as you will see in Worked Exercise F46.

The **interval of convergence** of the power series is the interval $(a - R, a + R)$, together with any endpoints of this interval at which the power series converges.

Figure 9 illustrates the various possible types of interval of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$.

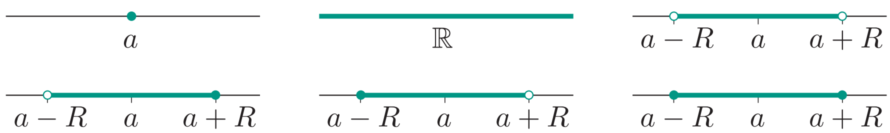


Figure 9 Possible types of intervals of convergence

Theorem F66 tells us that each power series has a radius of convergence R, but it does not tell us how to find R. However, a power series is a particular type of series, so the convergence tests for series from Unit D3 can be applied.

We can find the radius of convergence of many power series by using the following version of the Ratio Test for series from Subsection 2.1 of Unit D3.

Theorem F67 Ratio Test for power series

Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series with radius of convergence R, and that

$$\left|\frac{a_{n+1}}{a_n}\right| \to L \text{ as } n \to \infty.$$

(a) If L is ∞, then $R = 0$.

(b) If $L = 0$, then $R = \infty$.

(c) If $L > 0$, then $R = 1/L$.

Proof We give this proof only in the case that $a = 0$. The proof of the general case is similar. It follows from the statement about absolute convergence in Theorem F66 that it is sufficient to consider the convergence of the series $\sum_{n=0}^{\infty} |a_n x^n|$ as this is convergent precisely when $\sum_{n=0}^{\infty} a_n x^n$ is convergent. This enables us to base the proof on the Ratio Test for series which can only be applied to a series of positive terms.

(a) Suppose that

$$\left|\frac{a_{n+1}}{a_n}\right| \to \infty \text{ as } n \to \infty.$$

Then, for $x \neq 0$,

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \left|\frac{a_{n+1}}{a_n}\right| |x| \to \infty \text{ as } n \to \infty,$$

so $\sum_{n=0}^{\infty} |a_n x^n|$ is divergent, by the Ratio Test for series.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ converges only for $x = 0$, so $R = 0$.

(b) Now suppose that

$$\left|\frac{a_{n+1}}{a_n}\right| \to 0 \text{ as } n \to \infty.$$

Then, for $x \neq 0$,

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \left|\frac{a_{n+1}}{a_n}\right| |x| \to 0 \times |x| = 0 < 1 \text{ as } n \to \infty,$$

so $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent, by the Ratio Test for series.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$, so $R = \infty$.

(c) Finally, suppose that

$$\left|\frac{a_{n+1}}{a_n}\right| \to L \text{ as } n \to \infty,$$

where $L > 0$.

If $|x| > 1/L$, then

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \left|\frac{a_{n+1}}{a_n}\right| |x| \to L|x| > 1 \text{ as } n \to \infty,$$

so $\sum_{n=0}^{\infty} |a_n x^n|$ is divergent, by the Ratio Test for series. Thus if $|x| > L$, then the series $\sum_{n=0}^{\infty} a_n x^n$ is not absolutely convergent and hence is not convergent, so it follows that $R \leq 1/L$.

However, if $0 < |x| < 1/L$, then

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \left|\frac{a_{n+1}}{a_n}\right| |x| \to L|x| < 1 \text{ as } n \to \infty,$$

so $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent, by the Ratio Test for series. Thus if $|x| < L$, then the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent and hence, by Theorem F66, is convergent, from which it follows that $R \geq 1/L$.

Taken together, these results show that $R = 1/L$, which completes the proof. ■

Worked Exercise F45

Determine the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n(x-1)^n}{n!}$

Solution

(a) Here $a_n = 1/n!$, for $n = 0, 1, 2, \ldots$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)!} \times \frac{n!}{1} = \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

Hence, by the Ratio Test, $R = \infty$.

Thus this power series converges for all x.

(b) Here $a_n = n^n/n!$, for $n = 0, 1, 2, \ldots$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty.$$

You met this limit in Subsection 5.3 of Unit D2.

Hence, by the Ratio Test, the radius of convergence is $R = 1/e$.

Thus this power series converges for $|x-1| < 1/e$, and diverges for $|x-1| > 1/e$.

Exercise F63

Determine the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} (2^n + 4^n)x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ (c) $\displaystyle\sum_{n=0}^{\infty} (n + 2^{-n})(x-1)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

The next exercise concerns a power series which plays an important role in Section 4, where we will use it to prove a generalised binomial theorem. In this power series α can be any real number, but if $\alpha \in \{0, 1, 2, \ldots\}$ then the series has only finitely many non-zero terms and thus converges for all values of x. The exercise asks you to determine the radius of convergence of the power series for other values of α.

Exercise F64

Determine the radius of convergence of the power series

$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots,$$

where $\alpha \neq 0, 1, 2, \ldots$.

The Ratio Test gives an open interval on which a power series converges. To determine the full interval of convergence of a power series with finite non-zero radius of convergence, we need to use other tests to find the behaviour at the interval endpoints.

Strategy F13

To find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$, do the following.

1. Use the Ratio Test for power series to find the radius of convergence R.
2. If R is finite and non-zero, use other tests for series to determine the behaviour of the power series at the endpoints of the interval $(a-R, a+R)$.

You met a general strategy for applying the various tests for convergence or divergence of a series in Subsection 3.3 of Unit D3 (Strategy D13, which you can also find in the module Handbook). You may find it helpful to refer to this general strategy when applying step 2 of Strategy F13.

Note in particular that we can use Strategy F13 to establish the largest possible range of validity of a Taylor series. Since a Taylor series is a power series, its largest possible range of validity is the interval of convergence obtained by using Strategy F13.

In the following worked exercise and exercise, each of the power series has coefficient $a_0 = 0$, so the sum starts at $n = 1$.

Worked Exercise F46

Determine the interval of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n^2}$

Solution

In each case, we apply Strategy F13.

(a) Here $a_n = 1$, for $n = 1, 2, \ldots$.

1. Since
$$\left|\frac{a_{n+1}}{a_n}\right| = 1, \quad \text{for } n = 1, 2, \ldots,$$
we have $R = 1$, by the Ratio Test. Thus (by the Radius of Convergence Theorem) this power series
   - converges for $-1 < x < 1$,
   - diverges for $x > 1$ and $x < -1$.

2. If $x = 1$, then the power series is $\displaystyle\sum_{n=1}^{\infty} 1^n$, which is divergent by the Non-null Test.

   If $x = -1$, then the power series is $\displaystyle\sum_{n=1}^{\infty} (-1)^n$, which is also divergent by the Non-null Test.

Hence the interval of convergence is $(-1, 1)$.

(b) Here $a_n = 1/n$, for $n = 1, 2, \ldots$.

1. Since
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \times \frac{n}{1} = \frac{1}{1+1/n} \to 1 \text{ as } n \to \infty,$$
we have $R = 1$, by the Ratio Test. Thus this power series
   - converges for $-1 < x < 1$,
   - diverges for $x > 1$ and $x < -1$.

2. If $x = 1$, then the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which is a basic divergent series.

   If $x = -1$, then the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is convergent by the Alternating Test.

Hence the interval of convergence is $[-1, 1)$.

(c) Here $a_n = 1/(2^n n^2)$, for $n = 1, 2, \ldots$.

1. Since
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}(n+1)^2} \times \frac{2^n n^2}{1}$$
$$= \frac{1}{2(1+1/n)^2} \to \frac{1}{2} \text{ as } n \to \infty,$$
we have $R = 2$, by the Ratio Test. Since $a = 3$, this power series
   - converges for $1 < x < 5$,
   - diverges for $x > 5$ and $x < 1$.

   This follows because
$$|x-3| < 2 \iff -2 < x-3 < 2$$
$$\iff 1 < x < 5,$$
by the rules for rearranging inequalities.

2. If $x = 5$, then the power series is
$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}(5-3)^n = \sum_{n=1}^{\infty} \frac{1}{n^2},$$
which is a basic convergent series.
If $x = 1$, then the power series is
$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}(1-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$
which is convergent by the Absolute Convergence Test.

Hence the interval of convergence is $[1, 5]$.

Exercise F65

Determine the interval of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x-5)^n$

3.2 Proof of the Radius of Convergence Theorem (optional)

In what follows, we often use without reference the fact that an absolutely convergent series is convergent.

To prove the Radius of Convergence Theorem, we need the following preliminary result.

Lemma F68

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x_0 \neq 0$, then it is absolutely convergent on the interval $(-|x_0|, |x_0|)$.

Figure 10 illustrates the statement of the lemma in the two possible cases $x_0 > 0$ and $x_0 < 0$.

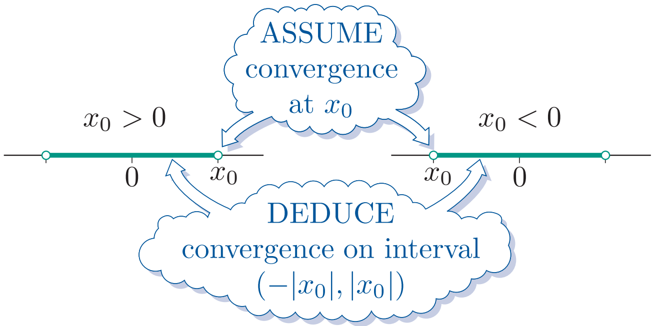


Figure 10 The two cases of Lemma F68

Proof of Lemma F68 First we write $r = |x_0|$. Since the series $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, the sequence $(a_n x_0^n)$ is null by Theorem D27 in Unit D3, and hence there is a number K such that

$$|a_n| r^n = |a_n x_0^n| \leq K, \quad \text{for } n = 0, 1, 2, \ldots. \tag{6}$$

Suppose that $|x| < r$. To prove that $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent, we write

$$a_n x^n = a_n r^n \left(\frac{x}{r}\right)^n.$$

Then, by inequality (6),

$$|a_n x^n| = |a_n| r^n \left|\frac{x}{r}\right|^n \leq K \left(\frac{|x|}{r}\right)^n.$$

Since $|x| < r$, we have $|x|/r < 1$, so the geometric series $\sum_{n=0}^{\infty} K \left(\frac{|x|}{r}\right)^n$ is convergent. Hence, by the Comparison Test (Theorem D30 in Unit D3), $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent, as required. ■

We can now give the proof of the Radius of Convergence Theorem, which we first state again for convenience.

Theorem F66 Radius of Convergence Theorem

For a given power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, exactly one of the following possibilities occurs.

(a) The series converges only for $x = a$.

(b) The series converges for all x.

(c) There is a number $R > 0$ such that

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges if } |x-a| < R$$

and

$$\sum_{n=0}^{\infty} a_n(x-a)^n \text{ diverges if } |x-a| > R.$$

Moreover, in parts (a), (b) and (c) the series converges absolutely on the specified sets of convergence.

Proof We give the proof only in the case $a = 0$. The proof of the general case is similar.

First we define the set

$$E = \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ is convergent} \right\}.$$

If $E = \{0\}$, then possibility (a) holds.

If E is unbounded, then for every $x \in \mathbb{R}$ there exists some $x_0 \in E$ such that $|x| < |x_0|$. Thus the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent, by Lemma F68. Since this is true for *every* $x \in \mathbb{R}$, it follows that possibility (b) holds.

Otherwise, the set E is bounded and contains a point $x_0 \neq 0$. Then $(-|x_0|, |x_0|) \subseteq E$, by Lemma F68, so $\sup E \geq |x_0|$. We define $R = \sup E$, where R is the radius of convergence.

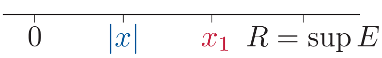


Figure 11 The case when $|x| < R$

If $|x| < R$, then we can find $x_1 \in E$ such that $|x| < x_1$; see Figure 11. Thus, by Lemma F68, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

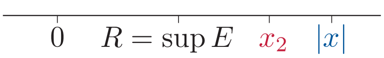


Figure 12 The case when $|x| > R$

If $|x| > R$, then we can find $x_2 > R$ such that $|x| > x_2$; see Figure 12. So $\sum_{n=0}^{\infty} a_n x^n$ is divergent (since if $\sum_{n=0}^{\infty} a_n x^n$ is convergent, then $\sum_{n=0}^{\infty} a_n x_2^n$ is convergent, by Lemma F68). Hence possibility (c) holds.

The above arguments show that in each case the given power series is not just convergent but *absolutely* convergent at each interior point of its interval of convergence.

This completes the proof. ■

4 Manipulating Taylor series

In Section 2 you saw that many functions can be represented by a Taylor series; if there exists $R > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

for $|x-a| < R$, then the expression on the right is the Taylor series at a for f.

In this section you will meet several rules which enable us to obtain 'new Taylor series from old', and so build on the list of basic Taylor series given in Theorem F65. Some of the rules for manipulating Taylor series are similar to the corresponding rules for continuous or differentiable functions.

We begin by noting that we can obtain new Taylor series from old by replacing x with another expression. For example, you have already seen the following Taylor series at 0:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1. \tag{7}$$

This power series has radius of convergence 1. Since $|-x| < 1$ if and only if $|x| < 1$, we can deduce from equation (7) that

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{for } |x| < 1,$$

and this power series also has radius of convergence 1.

Similarly, since $|3x^2| < 1$ if and only if $|x| < 1/\sqrt{3}$, we can deduce from equation (7) that

$$\begin{aligned}\frac{1}{1-3x^2} &= 1 + 3x^2 + (3x^2)^2 + (3x^2)^3 + \cdots \\ &= \sum_{n=0}^{\infty} 3^n x^{2n}, \quad \text{for } |x| < 1/\sqrt{3},\end{aligned}$$

and this power series has radius of convergence $1/\sqrt{3}$.

4.1 The Combination Rules and the Power Rule

We now study the Combination Rules for Taylor series.

Theorem F69 Combination Rules for Taylor series

Let f and g be functions that can both be represented by a Taylor series at a, and suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \text{ for } |x-a| < R,$$

$$g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n, \text{ for } |x-a| < R'.$$

Then the following hold for $r = \min\{R, R'\}$ and $\lambda \in \mathbb{R}$:

Sum Rule $$(f+g)(x) = \sum_{n=0}^{\infty}(a_n + b_n)(x-a)^n, \quad \text{for } |x-a| < r$$

Multiple Rule $$\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n(x-a)^n, \text{ for } |x-a| < R.$$

Remarks

1. The Sum and Multiple Rules for Taylor series are simply special cases of the Sum and Multiple Rules for general convergent series (see Theorem D25 in Unit D3.)
2. The radius of convergence of the Taylor series for $f+g$ may be larger than $r = \min\{R, R'\}$: Theorem F69 simply asserts that it must be *at least* r. For example, we can use the standard geometric series and the Sum and Multiple Rules to verify that the Taylor series at 0 for the functions $f(x) = \dfrac{1}{1-x}$ and $g(x) = \dfrac{-1}{1-x} + \dfrac{1}{1-x/2}$ are

   $$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

   and, by replacing x with $x/2$,

   $$\begin{aligned} g(x) &= -(1 + x + x^2 + x^3 + \cdots) + (1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + \cdots) \\ &= \sum_{n=0}^{\infty}\left(-1 + \frac{1}{2^n}\right)x^n. \end{aligned}$$

The Taylor series for each of f and g has radius of convergence 1, by the Ratio Test for power series, so Theorem F69 tells us that the radius of convergence of the Taylor series for $f+g$ is at least 1. However, the Taylor series for the function $(f+g)(x) = \dfrac{1}{1-x/2}$ is

$$(f+g)(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \tfrac{1}{8}x^3 + \tfrac{1}{16}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n} x^n,$$

which has radius of convergence 2, by the Ratio Test for power series.

Worked Exercise F47

Find the Taylor series at 0 for $f(x) = \cosh x$.

Solution

We can write

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \text{for } x \in \mathbb{R}.$$

We know that, for $x \in \mathbb{R}$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

This is one of the basic Taylor series given in Theorem F65.

and so, by replacing x with $-x$,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots (-1)^n \frac{x^n}{n!} + \cdots.$$

Then, by the Sum Rule, we deduce that

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots\right), \quad \text{for } x \in \mathbb{R},$$

since the odd-powered terms cancel. It then follows, by using the Multiple Rule with $\lambda = \frac{1}{2}$, that for $x \in \mathbb{R}$,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots.$$

Exercise F66

Find the Taylor series at 0 for each of the following functions.

(a) $f(x) = \sinh x$ (b) $f(x) = \log(1-x) + \dfrac{2}{1-x}$

We now look at how we might find the Taylor series at 0 for the function $f(x) = \dfrac{1+x}{1-x}$.

We can express $\dfrac{1+x}{1-x}$ in the form $(1+x) \times \dfrac{1}{1-x}$, so using the Taylor series for $\dfrac{1}{1-x}$ we have

$$\begin{aligned}\frac{1+x}{1-x} &= (1+x) \times \frac{1}{1-x}\\ &= (1+x) \times (1 + x + x^2 + x^3 + \cdots + x^n + \cdots), \text{ for } |x| < 1.\end{aligned}$$

If we then simply multiply out these two brackets and collect together the multiples of successive powers of x, we get

$$\begin{aligned}\frac{1+x}{1-x} &= 1 \times (1 + x + x^2 + x^3 + \cdots + x^n + \cdots)\\ &\quad + x \times (1 + x + x^2 + x^3 + \cdots + x^n + \cdots)\\ &= 1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots,\end{aligned}$$

and this power series has radius of convergence 1.

To justify multiplying together Taylor series in this way to obtain further Taylor series, we now state and prove the Product Rule for Taylor series. If you are short of time, then you may prefer to skim read the proof.

Theorem F70 Product Rule for Taylor series

Let f and g be functions that can both be represented by a Taylor series at a, and suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \quad \text{for } |x-a| < R,$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n, \quad \text{for } |x-a| < R'.$$

Then if $r = \min\{R, R'\}$ we have

$$(fg)(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for } |x-a| < r,$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

Note that although the expression for the coefficient c_n looks rather complicated, it is just the result of multiplying the two Taylor series term by term and summing all the resulting coefficients of $(x-a)^n$.

Proof of Theorem F70 For simplicity, we assume that $a = 0$.

Take $n \geq 2$ and put $m = \lfloor \frac{1}{2}n \rfloor$, the integer part of $\frac{1}{2}n$. Then

$$\sum_{i=0}^{n} a_i x^i \times \sum_{j=0}^{n} b_j x^j = \sum_{k=0}^{n} c_k x^k + \text{the sum of those terms } a_i b_j x^{i+j} \text{ in } \sum_{i=0}^{n} a_i x^i \times \sum_{j=0}^{n} b_j x^j \text{ with } i + j > n.$$

Thus, by the Triangle Inequality,

$$\left| \sum_{i=0}^{n} a_i x^i \times \sum_{j=0}^{n} b_j x^j - \sum_{k=0}^{n} c_k x^k \right| \leq \text{the sum of those terms } \left| a_i b_j x^{i+j} \right| \text{ in } \sum_{i=0}^{n} |a_i x^i| \times \sum_{j=0}^{n} |b_j x^j| \text{ with } i + j > n.$$

But all the latter terms are included in the expression

$$\sum_{i=0}^{n} |a_i x^i| \times \sum_{j=0}^{n} |b_j x^j| - \sum_{i=0}^{m} |a_i x^i| \times \sum_{j=0}^{m} |b_j x^j|,$$

and the other terms in this expression are all non-negative.

Note that all of the terms $\left| a_i b_j x^{i+j} \right|$ in $\sum_{i=0}^{m} |a_i x^i| \times \sum_{j=0}^{m} |b_j x^j|$ have $i + j \leq 2m \leq n$.

Hence

$$\left| \sum_{i=0}^{n} a_i x^i \times \sum_{j=0}^{n} b_j x^j - \sum_{k=0}^{n} c_k x^k \right| \leq \sum_{i=0}^{n} |a_i x^i| \times \sum_{j=0}^{n} |b_j x^j| - \sum_{i=0}^{m} |a_i x^i| \times \sum_{j=0}^{m} |b_j x^j|. \tag{8}$$

Now suppose that $|x| < r$, so the series $\sum_{i=0}^{\infty} |a_i x^i|$ and $\sum_{j=0}^{\infty} |b_j x^j|$ are both convergent, with sums s and t, respectively.

Here we have used the fact that a power series is absolutely convergent at each interior point of its interval of convergence.

As $n \to \infty$, the right-hand side of inequality (8) tends to $st - st = 0$, since $m \to \infty$. Thus, by the Limit Inequality Rule (Theorem D11 in Unit D2), $\sum_{k=0}^{\infty} c_k x^k$ converges, and

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{i=0}^{\infty} a_i x^i \times \sum_{j=0}^{\infty} b_j x^j.$$

This completes the proof of the Product Rule. ∎

Worked Exercise F48

Find the Taylor series at 0 for the function $f(x) = \dfrac{1+x}{(1-x)^2}$.

Solution

We first write

$$\frac{1+x}{(1-x)^2} = \frac{1+x}{1-x} \times \frac{1}{1-x}.$$

We know that, for $|x| < 1$,

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

We discussed the first of these two series just before stating the Product Rule, and our informal argument there is justified by the proof of the Product Rule. The second series is a basic Taylor series listed in Theorem F65.

Hence, by the Product Rule,

$$\begin{aligned}\frac{1+x}{(1-x)^2} &= \frac{1+x}{1-x} \times \frac{1}{1-x} \\ &= (1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots) \\ &\quad \times (1 + x + x^2 + x^3 + \cdots + x^n + \cdots) \\ &= 1 + (2+1)x + (2+2+1)x^2 + \cdots \\ &\quad + (2+2+\cdots+2+1)x^n + \cdots \\ &= 1 + 3x + 5x^2 + \cdots + (2n+1)x^n + \cdots,\end{aligned}$$

for $|x| < 1$.

Thus the Taylor series for f at 0 is $\displaystyle\sum_{n=0}^{\infty}(2n+1)x^n$, for $|x| < 1$.

Exercise F67

Determine the Taylor series at 0 for each of the following functions.

(a) $f(x) = (1+x)\log(1+x)$ (b) $f(x) = \dfrac{1}{(1-x)^2}$

(c) $f(x) = \dfrac{1+x}{(1-x)^3}$

Hint: In part (c), use the solution to Worked Exercise F48.

Exercise F68

Determine the Taylor series at 0 for each of the following functions. In each case, indicate the general term and state a range of validity for the series.

(a) $f(x) = \sinh x + \sin x$ (b) $f(x) = \dfrac{1}{1+2x^2}$

Hint: In part (a), use the solution to Exercise F66(a).

Exercise F69

Determine the first three non-zero terms in the Taylor series at 0 for the function $f(x) = e^x(1-x)^{-2}$, and state a range of validity for the series.

Hint: Use the solution to Exercise F67(b).

4.2 The Differentiation and Integration Rules

We have seen that the hyperbolic functions have the following Taylor series at 0:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{and} \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$

each with radius of convergence ∞. Notice that the derivative of the function $\sinh x$ is $\cosh x$, and that term-by-term differentiation of the series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ gives the series $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$. It looks as though we can obtain the Taylor series for the derivative of a function f simply by differentiating the Taylor series for f itself.

Our next two results show that we can differentiate or integrate the Taylor series of a function f term-by-term to obtain the Taylor series of the corresponding function f' or $\displaystyle\int f$, respectively, where we make an appropriate choice for the constant of integration.

Theorem F71 Differentiation Rule for Taylor series

The Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

have the same radius of convergence, R say.

Also, $f(x)$ is differentiable on $(a-R, a+R)$, and

$$f'(x) = g(x), \quad \text{for } |x-a| < R.$$

Note that the two series in the Differentiation Rule may behave differently at the endpoints of their respective intervals of convergence. For example, the power series

$$\frac{1}{1^2}x + \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}x^n$$

and

$$\frac{1}{1^2} + \frac{1}{2^2}2x + \frac{1}{3^2}3x^2 + \cdots = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}x^{n-1}$$

both have radius of convergence 1. However, the first series converges at both ± 1, whereas the second series converges at -1 but diverges at 1.

Proof of the Differentiation Rule (optional) For simplicity, we assume that $a = 0$.

Let the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have radii of convergence R and R', respectively. We prove that $R' = R$.

We first show that $R' \geq R$. To prove this, suppose that $|x| < R$. Now choose r such that $|x| < r < R$, as shown in Figure 13. Then $\sum_{n=0}^{\infty} a_n r^n$ is convergent, so $(a_n r^n)$ is a null sequence. Thus there is a positive number K such that

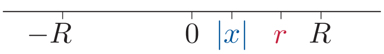


Figure 13 The relationship between r, $|x|$ and R.

$$|a_n r^n| \leq K, \quad \text{for } n = 0, 1, 2, \ldots. \tag{9}$$

Then

$$|n a_n x^{n-1}| = \frac{|n a_n r^n x^{n-1}|}{r^n} \leq \frac{K}{r}\, n \left(\frac{|x|}{r}\right)^{n-1}, \quad \text{for } n = 1, 2, \ldots, \tag{10}$$

by inequalities (9). Since $|x|/r < 1$, the series $\sum_{n=1}^{\infty} n\,(|x|/r)^{n-1}$ converges.

This follows from the solution to Exercise F65(a).

Therefore, by statement (10) and the Comparison Test (Theorem D30 in Unit D3), $\sum_{n=1}^{\infty} |n a_n x^{n-1}|$ is convergent for all $|x| < R$. This proves that $R' \geq R$.

Next we show that $R \geq R'$. To prove this, suppose that $|x| < R'$. Then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is absolutely convergent and

$$|a_n x^n| = |n a_n x^{n-1}| \frac{|x|}{n} \leq |x|\,|n a_n x^{n-1}|, \quad \text{for } n = 1, 2, \ldots.$$

Therefore, by the Comparison Test, $\sum_{n=1}^{\infty} |a_n x^n|$ is convergent for all $|x| < R'$. Thus $R \geq R'$, so we deduce that $R' = R$.

Differentiating the terms of $\sum_{n=1}^{\infty} na_n x^{n-1}$, we deduce that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \text{ also has radius of convergence } R, \tag{11}$$

by an analogous argument to the one above.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We now use statement (11) to prove that f' exists and has the required form on $(-R, R)$.

Take $x \in (-R, R)$, and choose r such that $|x| < r < R$. Then, for all h such that $|x+h| < r$ (see Figure 14),

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1} &= \sum_{n=1}^{\infty} \frac{a_n(x+h)^n}{h} - \sum_{n=1}^{\infty} \frac{a_n x^n}{h} - \sum_{n=1}^{\infty} na_n x^{n-1} \\ &= \sum_{n=2}^{\infty} a_n \frac{(x+h)^n - x^n - nx^{n-1}h}{h}. \qquad (12)\end{aligned}$$

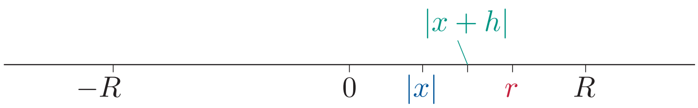


Figure 14 The point $|x+h|$

Now we apply Taylor's Theorem to the function $p(x) = x^n$ on an open interval containing x and $x+h$. We obtain

$$p(x+h) = (x+h)^n = x^n + nx^{n-1}h + \frac{1}{2!}n(n-1)c_n^{n-2}h^2,$$

where c_n lies between x and $x+h$.

By Taylor's Theorem, we have

$$p(x+h) = p(x) + p'(x)((x+h)-x) + R_1(x),$$

where $p'(x) = nx^{n-1}$ and the remainder term $R_1(x)$ depends on the second derivative $p''(x) = n(n-1)x^{n-2}$.

Then $|c_n| < r$, so

$$|(x+h)^n - x^n - nx^{n-1}h| \leq \tfrac{1}{2}n(n-1)r^{n-2}|h|^2. \tag{13}$$

By equation (12) and inequality (13), together with the Triangle Inequality, we obtain

$$\left| \frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1} \right| \leq \tfrac{1}{2}|h| \sum_{n=2}^{\infty} n(n-1)|a_n| r^{n-2}. \tag{14}$$

Since $r < R$, the series $\sum_{n=2}^{\infty} n(n-1)a_n r^{n-2}$ is absolutely convergent, by statement (11). Thus, by inequality (14) and the Limit Inequality Rule,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad \text{for } |x| < R.$$

We can now easily obtain the Integration Rule from the Differentiation Rule.

Theorem F72 Integration Rule for Taylor series

The Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{and} \quad F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

have the same radius of convergence, R say.

Also, if $R > 0$, then

$$\int f(x)\,dx = F(x), \quad \text{for } |x-a| < R.$$

Remarks

1. As in the Differentiation Rule, the two series in the theorem may behave differently at the endpoints of their respective intervals of convergence.
2. The final conclusion says that F is a primitive of f on $(a-R, a+R)$. It is sometimes expressed in the following way:

   $$\int \left(\sum_{n=0}^{\infty} a_n (x-a)^n \right) dx = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

   We find c by putting $x = a$ into the equation.

Proof of the Integration Rule The two series have the same radius of convergence, by the Differentiation Rule applied to F.

By the same rule, $F' = f$, so F is a primitive of f on $(a-R, a+R)$. ■

Worked Exercise F49

Find the Taylor series at 0 for $f(x) = \tan^{-1} x$.

Solution

We know that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \quad \text{for } x \in \mathbb{R}.$$

We also know that, for $|x| < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

You can see this by replacing x by $-x^2$ in the basic series for $1/(1-x)$.

Hence, by the Integration Rule,

$$\tan^{-1} x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1},$$

for $|x| < 1$, where c is a constant.

Substituting $x = 0$ into this equation, we find that $c = \tan^{-1} 0 = 0$. Hence

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots,$$

for $|x| < 1$.

In fact the above formula also holds for $x = 1$ and $x = -1$, but the proof of these facts is somewhat more complicated and we do not give it here.

Exercise F70

Find the Taylor series at 0 for:

(a) $f(x) = (1-x)^{-3}$ (b) $f(x) = \tanh^{-1} x$.

Hint: You may find the solution to Exercise F67(b) helpful in part (a).

Exercise F71

Determine the Taylor series at 0 for the function $f(x) = e^{-x^2}$. Deduce that

$$\int_0^1 e^{-x^2}\, dx = 1 - \frac{1}{3} + \frac{1}{10} - \cdots + \frac{(-1)^n}{(2n+1)n!} + \cdots.$$

Exercise F72

Use the Taylor series for $1/(1+x)$ at 0 to determine the Taylor series for the function $g(x) = \log(1+x)$ at the same point, and state a range of validity for this series.

In Exercise F72 you established that the Taylor series at 0 for the function $f(x) = \log(1+x)$ is valid on the interval $(-1, 1)$. In Section 2 we proved that this Taylor series is valid on $[0, 1]$, so by combining these results we see that the full range of validity for the Taylor series at 0 for the function $f(x) = \log(1+x)$ is the interval $(-1, 1]$, as stated in Theorem F65(e).

4.3 The General Binomial Theorem and the Uniqueness Theorem

In Subsection 3.4 of Unit D1 *Numbers* you met the Binomial Theorem, which states that, for each positive integer n,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{for } k \in \mathbb{N}.$$

This gives the Taylor series for $(1+x)^n$ and is valid for all $x \in \mathbb{R}$. In fact, a similar result known as the *General Binomial Theorem* holds for more general powers of $(1+x)$, but with a restriction on the values of x for which it is valid.

Theorem F73 General Binomial Theorem

For $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \text{for } |x| < 1,$$

where

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad \text{for } n \in \mathbb{N}.$$

Remarks

1. The Binomial Theorem is usually stated as a sum of powers of x^k. We have stated the General Binomial Theorem as a sum of powers of x^n in order to match the Taylor series in the rest of the unit. Note that this means that the role of n in the generalised binomial coefficient is the same as the role of k in the normal binomial coefficient.
2. The coefficient $\binom{\alpha}{n}$ is known as a **generalised binomial coefficient**. An example of calculating a generalised binomial coefficient is

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} \\ &= (-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n n!}.\end{aligned}$$

We now give the proof of the General Binomial Theorem. If you are short of time, then you may wish to skim read this proof.

Proof of the General Binomial Theorem

Let

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad \text{and} \quad g(x) = f(x)(1+x)^{-\alpha}, \qquad \text{for } |x| < 1.$$

We first note that the series $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ converges for $|x| < 1$, as proved in Exercise F64. We want to prove that $g(x) = 1$ and hence that $f(x) = (1+x)^{\alpha}$, for all x with $|x| < 1$.

Using the Product Rule for differentiation, we differentiate the expression for g to obtain

$$\begin{aligned} g'(x) &= f'(x)(1+x)^{-\alpha} - \alpha f(x)(1+x)^{-\alpha-1} \\ &= ((1+x)f'(x) - \alpha f(x))(1+x)^{-\alpha-1}. \end{aligned} \tag{15}$$

Now, using the Differentiation Rule,

$$\begin{aligned} (1+x)f'(x) - \alpha f(x) &= (1+x)\sum_{n=1}^{\infty} n\binom{\alpha}{n} x^{n-1} - \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= \sum_{n=1}^{\infty} n\binom{\alpha}{n} x^{n-1} + \sum_{n=1}^{\infty} n\binom{\alpha}{n} x^n - \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \\ &= \sum_{n=0}^{\infty} (n+1)\binom{\alpha}{n+1} x^n + \sum_{n=0}^{\infty} (n-\alpha)\binom{\alpha}{n} x^n, \end{aligned}$$

since

$$\sum_{n=1}^{\infty} n\binom{\alpha}{n} x^{n-1} = \sum_{n=0}^{\infty} (n+1)\binom{\alpha}{n+1} x^n,$$

and

$$\sum_{n=1}^{\infty} n\binom{\alpha}{n} x^n = \sum_{n=0}^{\infty} n\binom{\alpha}{n} x^n.$$

Hence

$$(1+x)f'(x) - \alpha f(x) = \sum_{n=0}^{\infty} \left((n+1)\binom{\alpha}{n+1} + (n-\alpha)\binom{\alpha}{n} \right) x^n. \tag{16}$$

We now use algebraic manipulation and the definition of the generalised binomial coefficient to simplfy the expression in square brackets in equation (16).

$$\begin{aligned}(n+1)\binom{\alpha}{n+1}+(n-\alpha)\binom{\alpha}{n} &= (n+1)\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{(n+1)!} \\ &\quad+(n-\alpha)\binom{\alpha}{n} \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(\alpha-n) \\ &\quad+(n-\alpha)\binom{\alpha}{n} \\ &= \binom{\alpha}{n}(\alpha-n)+(n-\alpha)\binom{\alpha}{n} \\ &= 0.\end{aligned}$$

Working backwards, it follows from equation (16) that $(1+x)f'(x)-\alpha f(x)=0$, and then from equation (15) that $g'(x)=0$.

So, $g(x)$ is a constant. Hence

$$\begin{aligned}g(x)=g(0)&=f(0)(1+0)^{-\alpha} \\ &=f(0)=\binom{\alpha}{0}=1,\end{aligned}$$

as required. ■

Worked Exercise F50

Use the General Binomial Theorem to find the first three non-zero terms in the Taylor series at 0 for the function $f(x)=(1+2x)^{-6}$.

Solution

By the General Binomial Theorem,

$$(1+2x)^{-6}=\sum_{n=0}^{\infty}\binom{-6}{n}(2x)^n,\quad \text{for } |2x|<1,$$

where

$$\binom{-6}{0}=1 \quad\text{and}\quad \binom{-6}{n}=\frac{(-6)(-7)\cdots(-6-n+1)}{n!},\quad \text{for } n\in\mathbb{N}.$$

Hence

$$(1+2x)^{-6}=1+\frac{(-6)}{1}2x+\frac{(-6)(-7)}{2}(2x)^2+\cdots,\quad \text{for } |2x|<1.$$

So

$$(1+2x)^{-6}=1-12x+84x^2-\cdots,\quad \text{for } |x|<\tfrac{1}{2}.$$

Exercise F73

Use the General Binomial Theorem to find the first three non-zero terms in the Taylor series at 0 for the function $f(x) = (1 + 4x)^{-1/3}$.

We now have a variety of techniques for finding Taylor series:

- Taylor's Theorem
- the Combination Rules
- the Product Rule
- the Differentiation and Integration Rules
- the General Binomial Theorem.

But how do we know that these different techniques will always give us the same expression for the Taylor series of a given function? To end this section we prove a result which states that there is only one Taylor series for a function f at a given point a. Thus any valid method gives the same Taylor coefficients.

Theorem F74 Uniqueness Theorem for Taylor series

If

$$\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n, \quad \text{for } |x-a| < R,$$

then $a_n = b_n$, for $n = 0, 1, 2, \ldots$.

Proof Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n, \quad \text{for } |x-a| < R.$$

If we differentiate both equations n times using the Differentiation Rule, and put $x = a$, then we obtain

$$f^{(n)}(a) = (n!)a_n \quad \text{and} \quad g^{(n)}(a) = (n!)b_n.$$

Since $f(x) = g(x)$ for $|x-a| < R$, it follows that $f^{(n)}(a) = g^{(n)}(a)$. Hence $a_n = b_n$, for all $n = 0, 1, 2, \ldots$. ■

5 Numerical estimates for π

One of the problems that has fascinated mathematicians for thousands of years has been how to determine accurately various important irrational numbers such as $\sqrt{2}$, π and e. In this section you will see the role of Taylor series in the numerical estimation of π, and meet an ingenious proof that π is irrational.

Historical estimates for π

The use of the symbol π to denote the ratio of the circumference of a circle to its diameter is relatively new, being first introduced by William Jones in 1706. Its use was popularised by Leonhard Euler who employed it in his *Introductio in Analysin Infinitorum* of 1748.

Early estimates for π include the following:

Mesopotamia (c.2000 BCE) In 1936 a Babylonian clay tablet was excavated which gives the ratio of the perimeter of a regular hexagon to the circumference of the circumscribed circle as $\frac{57}{60} + \frac{36}{(60)^2}$ (the Babylonians used the sexagesimal (base 60) system rather than the decimal system). This corresponds to a value for π of 3.125.

Egypt (c.1900 BCE) The Egyptians found by experience that they could approximate the area of a circle with diameter d by reducing d by one-ninth and squaring it. This corresponds to a value for π of $\frac{256}{81} \approx 3.1605$.

Archimedes (c.250 BCE) Archimedes (c.287–c.212 BCE) considered hexagons inside and outside a circle and compared their perimeters with the circumference of the circle. This gave him a value for π between 3 and 3.464. He then replaced the hexagon with a 12-sided polygon and recalculated the lengths. He continued by progressively doubling the sides of the polygon until he reached a polygon of 96 sides. This gave him a value for π between $3\frac{10}{71}$ and $3\frac{1}{7}$, or $3.14084 < \pi < 3.14286$, which is correct to two decimal places. Archimedes' method was described in some detail in Subsection 5.2 of Unit D2.

China (c.100–500) Zhang Heng (78–139) considered the ratio of the area of a square to the area of a circle and the ratio of the volume of a cube to the volume of a sphere, and was led to an approximation for π of $\sqrt{10}\,(\approx 3.162)$. He also calculated π as $\frac{736}{232}\,(\approx 3.1724)$. Liu Hui (c.220–c.280) used a polygonal method similar to that of Archimedes. He doubled the sides of a regular polygon until he reached a polygon with 3072 sides, from which he calculated a value for π of 3.14159. Zu Chongzhi (429–500) deduced that $3.1415926 < \pi < 3.1415927$, which was the most accurate approximation for π for almost a millennium. It is not known how he obtained this result but it is possible that he considered polygons with $24\,576\,(= 2^{13} \times 3)$ sides.

India (499) Aryabhata (476–550) in his *Aryabhatiya* included the following statement: 'Add 4 to 100, multiply by 8 and add 62 000. The result is approximately the circumference of a circle of which the diameter is 20 000.' This gives an approximate value for π of 3.1416. Although Aryabhata does not say how this value was found, it is likely that it was done by the method of polygons, using polygons with 384 sides.

With the development of calculus in the seventeenth century, new formulas for estimating π were discovered, including Wallis' Formula which you met in Subsection 3.2 of Unit F3 *Integration*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots.$$

5.1 Tangent formulas

In Worked Exercise F49 you saw that the Taylor series at 0 for the function $\tan^{-1}$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad \text{for } x \in [-1, 1].$$

In particular, with $x = 1$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

The first of these two series is known as *Leibniz's series* and the second as *Gregory's series*, although these names are often interchanged.

The second series is not very useful for calculating π, as its successive partial sums converge far too slowly. The smaller the value of x, the faster the Taylor series for $\tan^{-1} x$ converges, so fewer terms are needed to calculate its sum to a given accuracy.

To obtain series that are more effective for calculating π, we can use an addition formula for $\tan^{-1}$. To derive this formula, first recall the addition formula for tan (given in the module Handbook):

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

If we now put $x = \tan a$ and $y = \tan b$, we have

$$\tan(a+b) = \frac{x+y}{1-xy},$$

so it follows that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \quad \text{for } x, y \in \mathbb{R}, \tag{17}$$

provided that $\tan^{-1} x + \tan^{-1} y$ lies in $(-\pi/2, \pi/2)$, the image set of $\tan^{-1}$.

For example, applying formula (17) with $x = \frac{1}{2}$ and $y = \frac{1}{3}$, we obtain

$$\tan^{-1}\left(\tfrac{1}{2}\right) + \tan^{-1}\left(\tfrac{1}{3}\right) = \tan^{-1} 1 = \pi/4.$$

Using the addition formula (17) with small values of x combined with the Taylor series for $\tan^{-1}$ at 0 gives efficient ways of calculating π. For example, repeated application of the formula gives the following (we omit the details):

$$4\tan^{-1}\left(\tfrac{1}{5}\right) - \tan^{-1}\left(\tfrac{1}{239}\right) = \pi/4,$$
$$6\tan^{-1}\left(\tfrac{1}{8}\right) + 2\tan^{-1}\left(\tfrac{1}{57}\right) + \tan^{-1}\left(\tfrac{1}{239}\right) = \pi/4.$$

The first of these formulas is called Machin's Formula. John Machin (1680–1751) used the formula to calculate the first 100 decimal places of π, and in 1974 such formulas were used to calculate π to a million decimal places. More recently, highly ingenious methods (based on techniques due to Gauss for evaluating integrals approximately) have been used to calculate π correct to many billions of decimal places.

For your interest, we now list the first 1000 decimal places of π.

3.1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679
8214808651 3282306647 0938446095 5058223172 5359408128
4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091
4564856692 3460348610 4543266482 1339360726 0249141273
7245870066 0631558817 4881520920 9628292540 9171536436
7892590360 0113305305 4882046652 1384146951 9415116094
3305727036 5759591953 0921861173 8193261179 3105118548
0744623799 6274956735 1885752724 8912279381 8301194912
9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132
0005681271 4526356082 7785771342 7577896091 7363717872
1468440901 2249534301 4654958537 1050792279 6892589235
4201995611 2129021960 8640344181 5981362977 4771309960
5187072113 4999999837 2978049951 0597317328 1609631859
5024459455 3469083026 4252230825 3344685035 2619311881
7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778
1857780532 1712268066 1300192787 6611195909 2164201989

There are several mnemonics that can be used to recall the first few digits of π, with the word lengths giving the successive digits of π. For example, one such mnemonic is:

May	I	have	a	large	container	of	coffee?
3.	1	4	1	5	9	2	6

5.2 Proof that π is irrational (optional)

We finish the unit with an interesting proof which uses several ideas that you have met in earlier analysis units. You began your study of analysis in this module with Unit D1 where you first considered rational and irrational numbers, which together make up the real numbers whose properties lie at the foundation of analysis. Now we come full circle and use some of the tools of analysis that you have learned to prove that π is irrational.

The first proof that π is irrational was given by Johann Heinrich Lambert in 1766, but the elegant, shorter proof that we give here was found by Ivan Niven in 1947.

Theorem F75

The number π is irrational.

Proof We prove that π^2 is irrational, from which it follows that π is irrational. The proof is by contradiction and the method is rather unusual, so we begin by outlining the two major steps.

First we show that if f is any polynomial function such that

$$0 < f''(x) + \pi^2 f(x) < 1, \quad \text{for } 0 < x < 1, \tag{18}$$

then

$$0 < f(0) + f(1) < \frac{1}{\pi}. \tag{19}$$

Next we show that if $\pi^2 = a/b$, for $a, b \in \mathbb{N}$, then there is a polynomial function f such that statement (18) is true but inequalities (19) do not hold. This contradiction shows that π^2 must be irrational.

Let f be a polynomial function satisfying statement (18), and put

$$g(x) = f'(x)\sin \pi x - \pi f(x)\cos \pi x.$$

Then

$$\begin{aligned} g'(x) &= f''(x)\sin \pi x + f'(x)\pi \cos \pi x - \pi f'(x)\cos \pi x + \pi^2 f(x)\sin \pi x \\ &= \left(f''(x) + \pi^2 f(x)\right)\sin \pi x. \end{aligned}$$

By the Mean Value Theorem (Theorem F37 in Unit F2), there exists $c \in (0, 1)$ such that

$$g(1) - g(0) = g'(c) = \left(f''(c) + \pi^2 f(c)\right)\sin \pi c.$$

Hence $0 < g(1) - g(0) < 1$, by statement (18) and the fact that $0 < \sin \pi c \leq 1$. But

$$g(1) - g(0) = \pi(f(0) + f(1)),$$

by the definition of g, so statement (19) follows.

Now suppose that $\pi^2 = a/b$, where $a, b \in \mathbb{N}$. Take $N \in \mathbb{N}$ so large that $\pi^2 a^N/N! < 1$,

This is possible because $(a^n/n!)$ is a basic null sequence: see Theorem D7 in Unit D2.

and put

$$p(x) = \frac{1}{N!} x^N (1-x)^N \tag{20}$$
$$= \frac{1}{N!}\left(c_N x^N + c_{N+1}x^{N+1} + \cdots + c_{2N}x^{2N}\right),$$

where the coefficients c_k are integers, for $N \leq k \leq 2N$. Then we have

$$0 < p(x) < 1/N!, \quad \text{for } 0 < x < 1, \tag{21}$$

by equation (20). Also, for $k = 0, 1, \ldots,$

$$p^{(k)}(0) = \begin{cases} 0, & 0 \leq k < N,\ k > 2N, \\ \dfrac{c_k k!}{N!}, & N \leq k \leq 2N, \end{cases}$$

so $p^{(k)}(0)$ is an integer. Hence $p^{(k)}(1)$ is also an integer, by the symmetry of the function p under the change of variable $x' = 1 - x$.

Now consider the polynomial function

$$f(x) = a^N p(x) - a^{N-1} b\, p^{(2)}(x) + \cdots + (-1)^N b^N p^{(2N)}(x),$$

which has degree $2N$. Then $f(0)$ and $f(1)$ are both integers, so statement (19) is false, since we know that $\pi > 1$. Finally,

$$f''(x) = a^N p^{(2)}(x) - a^{N-1} b\, p^{(4)}(x) + \cdots + (-1)^{N-1} a b^{N-1} p^{(2N)}(x),$$

since $p^{(2N+2)} = 0$. Hence

$$f''(x) + \pi^2 f(x) = f''(x) + \frac{a}{b} f(x) = \frac{a}{b} a^N p(x) = \pi^2 a^N p(x),$$

by telescopic cancellation. Thus, by statement (21) and our choice of N,

$$0 < f''(x) + \pi^2 f(x) < \pi^2 a^N/N! < 1, \quad \text{for } 0 < x < 1,$$

so statement (18) does hold. This completes the proof. ■

Summary

In this unit you have seen that many functions can be approximated by Taylor polynomials: for a function f that is n-times differentiable on an open interval containing the point a, the Taylor polynomial of degree n at a for f is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

You also saw how to estimate the accuracy of this approximation using Taylor's Theorem and an upper bound for a remainder term $R_n(x)$ that depends on the $(n+1)$th derivative of f. For a function f that has derivatives of all orders on an open interval, you learned that at points where the remainder term tends to zero as $n \to \infty$, f can be represented by a convergent power series known as the Taylor series for f.

You went on to study how functions can be defined by means of power series, and how the Ratio Test can be used to find their radius of convergence. You saw how to manipulate Taylor series to obtain new series from old by using the Combination Rules, the Product Rule and the Integration and Differentiation Rules. You also met the General Binomial Theorem, which gives the Taylor series at 0 for the function $(1+x)^\alpha$ for $\alpha \in \mathbb{R}$. Finally, you looked at how the Taylor series at 0 for $\tan^{-1} x$ can be used to obtain estimates for the value of π.

Learning outcomes

After working through this unit, you should be able to:

- calculate the *Taylor polynomial* $T_n(x)$ at a given point a of a given function f
- appreciate that in many cases $T_n(x)$ gives a good approximation to $f(x)$ for x near the point a
- state and use Taylor's Theorem
- appreciate that a sequence of Taylor polynomials may or may not converge at a given point to the value of the function at that point
- state and use certain basic Taylor series
- state the Radius of Convergence Theorem
- determine the *radius of convergence* and the *interval of convergence* of certain power series
- state and use the Combination Rules, Product Rule, Differentiation Rule and Integration Rule for Taylor series
- state and use the General Binomial Theorem
- understand and use the Uniqueness Theorem for Taylor series.

Table of standard Taylor series

Function	Taylor series	Domain
$\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
$\log(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}x^n}{n}$	$-1 < x \leq 1$
e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$x \in \mathbb{R}$
$\sin x$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\cos x$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$(1+x)^\alpha$	$1 + \alpha x + \dfrac{\alpha(\alpha-1)}{2!}x^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$	$\lvert x\rvert < 1,\ \alpha \in \mathbb{R}$
$\sinh x$	$x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\cosh x$	$1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\tan^{-1} x$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1}$	$\lvert x\rvert \leq 1$

Solutions to exercises

Solution to Exercise F55

The tangent approximation to f at a is

$$f(x) \approx f(a) + f'(a)(x-a).$$

(a) We have

$$\begin{aligned} f(x) &= e^x, \quad f(2) = e^2 \\ f'(x) &= e^x, \quad f'(2) = e^2. \end{aligned}$$

Hence the tangent approximation to f at 2 is

$$e^x \approx e^2 + e^2(x-2) = e^2(x-1).$$

(b) We have

$$\begin{aligned} f(x) &= \cos x, \quad & f(0) = 1 \\ f'(x) &= -\sin x, \quad & f'(0) = 0. \end{aligned}$$

Hence the tangent approximation to f at 0 is

$$\cos x \approx 1 + 0(x-0) = 1.$$

Solution to Exercise F56

(a) We have

$$\begin{aligned} f(x) &= e^x, \quad & f(2) = e^2 \\ f'(x) &= e^x, \quad & f'(2) = e^2 \\ f''(x) &= e^x, \quad & f''(2) = e^2 \\ f^{(3)}(x) &= e^x, \quad & f^{(3)}(2) = e^2. \end{aligned}$$

Hence

$$T_1(x) = f(2) + f'(2)(x-2) = e^2 + e^2(x-2)$$

$$\begin{aligned} T_2(x) &= T_1(x) + \frac{f''(2)}{2!}(x-2)^2 \\ &= e^2 + e^2(x-2) + \tfrac{1}{2}e^2(x-2)^2 \end{aligned}$$

$$\begin{aligned} T_3(x) &= T_2(x) + \frac{f^{(3)}(2)}{3!}(x-2)^3 \\ &= e^2 + e^2(x-2) + \tfrac{1}{2}e^2(x-2)^2 \\ &\quad + \tfrac{1}{6}e^2(x-2)^3. \end{aligned}$$

(b) We have

$$\begin{aligned} f(x) &= \cos x, \quad & f(0) = 1 \\ f'(x) &= -\sin x, \quad & f'(0) = 0 \\ f''(x) &= -\cos x, \quad & f''(0) = -1 \\ f^{(3)}(x) &= \sin x, \quad & f^{(3)}(0) = 0. \end{aligned}$$

Hence

$$T_1(x) = f(0) + f'(0)x = 1$$

$$T_2(x) = T_1(x) + \frac{f''(0)}{2!}x^2 = 1 - \tfrac{1}{2}x^2$$

$$T_3(x) = T_2(x) + \frac{f^{(3)}(0)}{3!}x^3 = 1 - \tfrac{1}{2}x^2.$$

Solution to Exercise F57

The Taylor polynomial of degree 4 for f at a is

$$\begin{aligned} T_4(x) = f(a) + f'(a)(x-a) &+ \frac{f''(a)}{2!}(x-a)^2 \\ + \frac{f^{(3)}(a)}{3!}(x-a)^3 &+ \frac{f^{(4)}(a)}{4!}(x-a)^4. \end{aligned}$$

(a) We have

$$\begin{aligned} f(x) &= \log(1+x), \quad & f(0) = 0 \\ f'(x) &= 1/(1+x), \quad & f'(0) = 1 \\ f''(x) &= -1/(1+x)^2, \quad & f''(0) = -1 \\ f^{(3)}(x) &= 2/(1+x)^3, \quad & f^{(3)}(0) = 2 \\ f^{(4)}(x) &= -6/(1+x)^4, \quad & f^{(4)}(0) = -6. \end{aligned}$$

Hence

$$T_4(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4.$$

(b) We have

$$\begin{aligned} f(x) &= \sin x, \quad & f(\pi/4) = 1/\sqrt{2} \\ f'(x) &= \cos x, \quad & f'(\pi/4) = 1/\sqrt{2} \\ f''(x) &= -\sin x, \quad & f''(\pi/4) = -1/\sqrt{2} \\ f^{(3)}(x) &= -\cos x \quad & f^{(3)}(\pi/4) = -1/\sqrt{2} \\ f^{(4)}(x) &= \sin x, \quad & f^{(4)}(\pi/4) = 1/\sqrt{2}. \end{aligned}$$

Hence

$$\begin{aligned} T_4(x) = &\frac{1}{\sqrt{2}}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2\right. \\ &\left. - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24}\left(x - \frac{\pi}{4}\right)^4\right). \end{aligned}$$

(c) Since $f(x) = 1 + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{4}x^4$, we have $f(0) = 1$ and

$$\begin{aligned} f'(x) &= \tfrac{1}{2} - x - \tfrac{1}{2}x^2 + x^3, & f'(0) &= \tfrac{1}{2} \\ f''(x) &= -1 - x + 3x^2, & f''(0) &= -1 \\ f^{(3)}(x) &= -1 + 6x, & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= 6, & f^{(4)}(0) &= 6. \end{aligned}$$

Hence

$$T_4(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{4}x^4.$$

(Note that $T_4(x) = f(x)$, as you might expect since f is a polynomial of degree 4.)

Solution to Exercise F58

From Worked Exercise F41(a), we have

$$T_3(x) = x - \tfrac{1}{6}x^3,$$

so

$$T_3(0.1) = 0.1 - 0.001/6 = 0.099\,8\overline{3}.$$

Since

$$\sin(0.1) = 0.099\,833\,416\ldots,$$

we have

$$\begin{aligned} |\sin(0.1) - T_3(0.1)| &= 0.099\,833\,416\ldots - 0.099\,8\overline{3} \\ &\leq 0.099\,833\,417 - 0.099\,833\,333 \\ &= 0.000\,000\,084 < 1 \times 10^{-7}, \end{aligned}$$

as required.

Solution to Exercise F59

(a) We have

$$f(x) = e^x, \quad f(0) = 1,$$

and in general, for each positive integer k,

$$f^{(k)}(x) = e^x, \quad f^{(k)}(0) = 1.$$

Hence

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

(b) We have

$$\begin{aligned} f(x) &= \log(1+x), & f(0) &= 0 \\ f'(x) &= 1/(1+x), & f'(0) &= 1 \\ f''(x) &= -1/(1+x)^2, & f''(0) &= -1 \\ f^{(3)}(x) &= 2/(1+x)^3, & f^{(3)}(0) &= 2 \end{aligned}$$

and in general, for each positive integer k,

$$f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(1+x)^k},$$

so that

$$f^{(k)}(0) = (-1)^{k+1}(k-1)!.$$

Hence

$$T_n(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots + (-1)^{n+1}\frac{x^n}{n}.$$

(c) We have

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1 \\ f'(x) &= -\sin x, & f'(0) &= 0 \\ f''(x) &= -\cos x, & f''(0) &= -1 \\ f^{(3)}(x) &= \sin x, & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= \cos x, & f^{(4)}(0) &= 1 \end{aligned}$$

and in general, for $k = 0, 1, 2, \ldots$,

$$f^{(2k)}(0) = (-1)^k \quad \text{and} \quad f^{(2k+1)}(0) = 0.$$

Hence, for $m = 0, 1, 2, \ldots$,

$$\begin{aligned} T_{2m}(x) &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!}, \end{aligned}$$

and $T_{2m+1}(x) = T_{2m}(x)$.

So if n takes either of the values $2m$ or $2m+1$, then we have

$$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!}.$$

Solution to Exercise F60

From Exercise F59(c), the nth Taylor polynomial at 0 for f is

$$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!},$$

where n is either $2m$ or $2m+1$. Hence, by Taylor's Theorem with $a = 0$, $f(x) = \cos x$ and $n = 3$,

$$\begin{aligned} \cos x &= T_3(x) + R_3(x) \\ &= 1 - \tfrac{1}{2}x^2 + R_3(x), \end{aligned}$$

where

$$R_3(x) = \frac{f^{(4)}(c)}{4!}\,x^4 = \frac{\cos c}{4!}\,x^4,$$

for some c between 0 and x, as required.

Solution to Exercise F61

(a) By the solution to Exercise F59(b), we have

$$T_2(x) = x - \tfrac{1}{2}x^2.$$

(b) We use Strategy F11 with $a = 0$, $x = 0.02$ and $n = 2$.

1. First, $f^{(3)}(x) = \dfrac{2}{(1+x)^3}$; see Exercise F59(b).
2. Thus
$$|f^{(3)}(c)| = \frac{2}{(1+c)^3} \leq 2, \quad \text{for } c \in [0, 0.2],$$
so we can take $M = 2$.
3. Hence
$$\begin{aligned} |\log(1.02) - T_2(0.02)| &= |R_2(0.02)| \\ &\leq \frac{M}{(2+1)!}|x-a|^{2+1} \\ &= \frac{2}{3!} \times |0.02 - 0|^3 \\ &= 0.000\,002\overline{6} \\ &< 3 \times 10^{-6}, \end{aligned}$$
as required.

(c) By part (a),

$$T_2(0.02) = 0.02 - \tfrac{1}{2}(0.02)^2 = 0.0198.$$

By part (b),

$$|\log(1.02) - T_2(0.02)| < 0.000\,003.$$

Hence

$$0.019\,797 < \log(1.02) < 0.019\,803,$$

so

$$\log(1.02) = 0.0198 \text{ (to 4 d.p.)}.$$

Solution to Exercise F62

(a) Using the derivatives of f found in the solution to Exercise F59(c), we obtain

$$f(\pi) = -1, \quad f'(\pi) = 0, \quad f''(\pi) = 1,$$
$$f^{(3)}(\pi) = 0, \quad f^{(4)}(\pi) = -1.$$

Hence

$$T_4(x) = -1 + \tfrac{1}{2}(x-\pi)^2 - \tfrac{1}{24}(x-\pi)^4.$$

(b) We use Strategy F12 with $I = [3\pi/4, 5\pi/4]$, $a = \pi$, $r = \pi/4$ and $n = 4$.

1. First, $f^{(5)}(x) = -\sin x$.
2. Thus
$$|f^{(5)}(c)| \leq 1, \quad \text{for } c \in [3\pi/4, 5\pi/4],$$
so we can take $M = 1$.
3. Hence
$$\begin{aligned} |R_4(x)| &\leq \frac{M}{(4+1)!} r^{4+1} \\ &= \frac{1}{5!}\left(\frac{\pi}{4}\right)^5 \\ &= 0.002\,49\ldots \\ &< 3 \times 10^{-3}, \quad \text{for } x \in [3\pi/4, 5\pi/4]. \end{aligned}$$

Thus $T_4(x)$ approximates $f(x)$ with an error less than 3×10^{-3} on $[3\pi/4, 5\pi/4]$.

Solution to Exercise F63

(a) Here $a_n = 2^n + 4^n$, for $n = 0, 1, 2, \ldots$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1} + 4^{n+1}}{2^n + 4^n}.$$

Dividing the numerator and denominator by 4^n gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2(1/2)^n + 4}{(1/2)^n + 1} \to 4 \text{ as } n \to \infty.$$

Hence, by the Ratio Test, the radius of convergence is $R = \frac{1}{4}$.

(b) Here $a_n = (n!)^2/(2n)!$, for $n = 1, 2, \ldots$, so

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{((n+1)!)^2}{(2n+2)!} \times \frac{(2n)!}{(n!)^2} \\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)}. \end{aligned}$$

Dividing the numerator and denominator by n^2 gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(1+1/n)(1+1/n)}{(2+2/n)(2+1/n)} \to \frac{1}{4} \text{ as } n \to \infty.$$

Hence, by the Ratio Test, the radius of convergence is $R = 4$.

(c) Here $a_n = n + 2^{-n}$, for $n = 0, 1, 2, \ldots$, so

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{n+1+2^{-n-1}}{n+2^{-n}} \\ &= \frac{1 + 1/n + 1/(n2^{n+1})}{1 + 1/(n2^n)} \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence, by the Ratio Test, the radius of convergence is $R = 1$.

(d) Here $a_n = n^n$, for $n = 1, 2, \dots$, so

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)^{(n+1)}}{n^n}\\ &= (n+1)\left(\frac{n+1}{n}\right)^n \to \infty \text{ as } n \to \infty.\end{aligned}$$

Hence, by the Ratio Test, the radius of convergence is $R = 0$; that is, the series converges only for $x = 0$.

Solution to Exercise F64

Applying the Ratio Test with

$$a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad \text{for } n \in \mathbb{N},$$

we obtain, for $\alpha \neq 0, 1, 2, \dots$,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{\alpha(\alpha-1)\cdots(\alpha-n+1)} \times \frac{n!}{(n+1)!}\right|\\ &= \left|\frac{\alpha-n}{n+1}\right|\\ &= \left|\frac{(\alpha/n)-1}{1+1/n}\right| \to 1 \text{ as } n \to \infty.\end{aligned}$$

Hence the radius of convergence is 1.

Solution to Exercise F65

In each case, we apply Strategy F13.

(a) Here $a_n = n$, for $n = 1, 2, \dots$.

1. Since
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{n+1}{n}\\ &= 1 + 1/n \to 1 \text{ as } n \to \infty,\end{aligned}$$
we have $R = 1$, by the Ratio Test. Thus this power series
    - converges for $-1 < x < 1$,
    - diverges for $x > 1$ and $x < -1$.
2. If $x = 1$, then the power series is
$$\sum_{n=0}^{\infty} n(1)^n = \sum_{n=0}^{\infty} n,$$
which is divergent, by the Non-null Test.
If $x = -1$, then the power series is
$$\sum_{n=0}^{\infty} n(-1)^n = \sum_{n=0}^{\infty} (-1)^n n,$$
which is again divergent, by the Non-null Test.

Hence the interval of convergence is $(-1, 1)$.

(b) Here $a_n = (-1)^n/(n3^n)$, for $n = 1, 2, \dots$.

1. Since
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{(n+1)3^{n+1}} \times \frac{n3^n}{1}\\ &= \frac{1}{(1+1/n)3} \to \frac{1}{3} \text{ as } n \to \infty,\end{aligned}$$
we have $R = 3$, by the Ratio Test. Since $a = 5$, this power series
    - converges for $2 < x < 8$,
    - diverges for $x > 8$ and $x < 2$.
2. If $x = 8$, then the power series is
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(8-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$
which is convergent, by the Alternating Test.
If $x = 2$, then the power series is
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(2-5)^n = \sum_{n=1}^{\infty} \frac{1}{n},$$
which is a basic divergent series.

Hence the interval of convergence is $(2, 8]$.

Solution to Exercise F66

(a) We know that, for $x \in \mathbb{R}$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots.$$

Hence, by the Sum and Multiple Rules,

$$\begin{aligned}\sinh x &= \tfrac{1}{2}(e^x - e^{-x})\\ &= x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots,\end{aligned}$$

for $x \in \mathbb{R}$.

(b) We know that, for $|x| < 1$,

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

Hence, by the Sum and Multiple Rules,

$$\begin{aligned}
&\log(1-x)+\frac{2}{1-x}\\
&=\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}-\cdots\right)\\
&\quad+(2+2x+2x^2+2x^3+\cdots+2x^n+\cdots)\\
&=2+x+\frac{3}{2}x^2+\frac{5}{3}x^3+\cdots+\left(2-\frac{1}{n}\right)x^n+\cdots,
\end{aligned}$$

for $|x|<1$.

Solution to Exercise F67

(a) We know that, for $|x|<1$,

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots.$$

Hence, by the Product Rule,

$$\begin{aligned}
&(1+x)\log(1+x)\\
&=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots\right)\\
&\quad+\left(x^2-\frac{x^3}{2}+\cdots+(-1)^n\frac{x^n}{n-1}+\cdots\right)\\
&=x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots+(-1)^n\frac{x^n}{n(n-1)}+\cdots,
\end{aligned}$$

for $|x|<1$.

(b) We know that, for $|x|<1$,

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots.$$

Hence, by the Product Rule,

$$\begin{aligned}
\frac{1}{(1-x)^2}&=\left(1+x+x^2+\cdots+x^n+\cdots\right)^2\\
&=1+(1+1)x+(1+1+1)x^2+\cdots\\
&\quad+(1+1+\cdots+1)x^n+\cdots\\
&=1+2x+3x^2+\cdots+(n+1)x^n+\cdots,
\end{aligned}$$

for $|x|<1$.

(c) We know that, for $|x|<1$,

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots,$$

and, for $|x|<1$, from Worked Exercise F48, that

$$\frac{1+x}{(1-x)^2}=1+3x+5x^2+\cdots+(2n+1)x^n+\cdots.$$

Hence, by the Product Rule,

$$\begin{aligned}
\frac{1+x}{(1-x)^3}&=(1+x+x^2+\cdots+x^n+\cdots)\\
&\quad\times(1+3x+5x^2+\cdots+(2n+1)x^n+\cdots)\\
&=1+(3+1)x+(5+3+1)x^2+\cdots\\
&\quad+((2n+1)+(2n-1)+\cdots+1)x^n+\cdots\\
&=1+4x+9x^2+\cdots+(n+1)^2x^n+\cdots,
\end{aligned}$$

for $|x|<1$, since $1+3+\cdots+(2n+1)$ is an arithmetic series with sum $(n+1)^2$.

Solution to Exercise F68

(a) We know from Exercise F66(a) and Theorem F65 that, for $x\in\mathbb{R}$,

$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

and

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots.$$

Hence, by the Sum Rule, for $x\in\mathbb{R}$,

$$\begin{aligned}
&\sinh x+\sin x\\
&=2x+\frac{2x^5}{5!}+\frac{2x^9}{9!}+\cdots+\frac{2x^{4n+1}}{(4n+1)!}+\cdots.
\end{aligned}$$

This Taylor series is valid for all real values of x.

(b) We know that

$$\frac{1}{1+x}=1-x+x^2-\cdots+(-1)^nx^n+\cdots,$$

for $|x|<1$. Replacing x by $2x^2$, we obtain

$$\frac{1}{1+2x^2}=1-2x^2+4x^4-\cdots+(-1)^n2^nx^{2n}+\cdots.$$

This last series converges for $2x^2<1$; that is, for $|x|<1/\sqrt{2}$.

Hence this Taylor series is valid in the interval $(-1/\sqrt{2},1/\sqrt{2})$.

Solution to Exercise F69

We know that

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots,\quad\text{for } x\in\mathbb{R}.$$

Also, for $|x|<1$, from Exercise F67(b),

$$(1-x)^{-2}=1+2x+3x^2+\cdots+(n+1)x^n+\cdots.$$

Hence, by the Product Rule,

$$\begin{aligned} &e^x(1-x)^{-2} \\ &= \left(1+x+\tfrac{1}{2}x^2+\cdots\right)\left(1+2x+3x^2+\cdots\right) \\ &= 1+(2+1)x+\left(3+2+\tfrac{1}{2}\right)x^2+\cdots \\ &= 1+3x+\tfrac{11}{2}x^2+\cdots, \quad \text{for } |x|<1. \end{aligned}$$

Solution to Exercise F70

(a) We know from the solution to Exercise F67(b) that, for $|x|<1$,

$$(1-x)^{-2} = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots.$$

Hence, by the Differentiation Rule,

$$2(1-x)^{-3} = 2+6x+\cdots+(n+1)nx^{n-1}+\cdots,$$

for $|x|<1$.

Applying the Multiple Rule, we obtain

$$(1-x)^{-3} = 1+3x+\cdots+\frac{(n+1)n}{2}x^{n-1}+\cdots,$$

for $|x|<1$.

(b) From the table of standard derivatives in the Quick reference section of the module Handbook, we have $f'(x)=\dfrac{1}{1-x^2}$ for $|x|<1$. Also, we know that

$$\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots, \quad \text{for } |x|<1.$$

Replacing x by x^2, we obtain

$$f'(x) = 1+x^2+x^4+\cdots+x^{2n}+\cdots, \quad \text{for } |x|<1.$$

Thus, by the Integration Rule, the Taylor series at 0 for f is

$$f(x) = c+x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots,$$

for $|x|<1$. Since $f(0)=0$, it follows that $c=0$.

Hence

$$\tanh^{-1}x = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots,$$

for $|x|<1$.

Solution to Exercise F71

Since

$$e^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots, \quad \text{for } x\in\mathbb{R},$$

we have

$$e^{-x^2} = 1-x^2+\frac{x^4}{2!}-\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots,$$

for $x\in\mathbb{R}$. It follows from the Integration Rule that

$$\begin{aligned} &\int_0^1 e^{-x^2}\,dx \\ &= \left[x-\frac{x^3}{3}+\frac{x^5}{5\times 2!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots\right]_0^1 \\ &= 1-\frac{1}{3}+\frac{1}{10}-\cdots+\frac{(-1)^n}{(2n+1)n!}+\cdots. \end{aligned}$$

Solution to Exercise F72

We know that

$$\frac{d}{dx}\log(1+x) = \frac{1}{1+x},$$

and that, for $|x|<1$,

$$\frac{1}{1+x} = 1-x+x^2-\cdots+(-1)^n x^n+\cdots.$$

Hence, by the Integration Rule,

$$\log(1+x) = c+x-\frac{x^2}{2}+\cdots+(-1)^n\frac{x^{n+1}}{n+1}+\cdots,$$

for $|x|<1$, where c is a constant.

On substituting $x=0$, we find that $c=0$. Hence

$$\log(1+x) = x-\frac{x^2}{2}+\cdots+(-1)^{n+1}\frac{x^n}{n}+\cdots,$$

for $|x|<1$. (Note that $(-1)^{n+1}=(-1)^{n-1}$.)

Solution to Exercise F73

By the General Binomial Theorem,

$$(1+4x)^{-1/3} = \sum_{n=0}^{\infty}\binom{-\frac{1}{3}}{n}(4x)^n, \quad \text{for } |4x|<1,$$

where $\dbinom{-\frac{1}{3}}{0}=1$ and

$$\binom{-\frac{1}{3}}{n} = \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\cdots\left(-\frac{1}{3}-n+1\right)}{n!}, \quad \text{for } n\in\mathbb{N}.$$

Hence

$$\begin{aligned} (1+4x)^{-1/3} &= 1+\frac{\left(-\frac{1}{3}\right)}{1}4x+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}(4x)^2+\cdots \\ &= 1-\tfrac{4}{3}x+\tfrac{32}{9}x^2-\cdots, \quad \text{for } |x|<\tfrac{1}{4}. \end{aligned}$$